AP Calculus AB

Greetings!!!

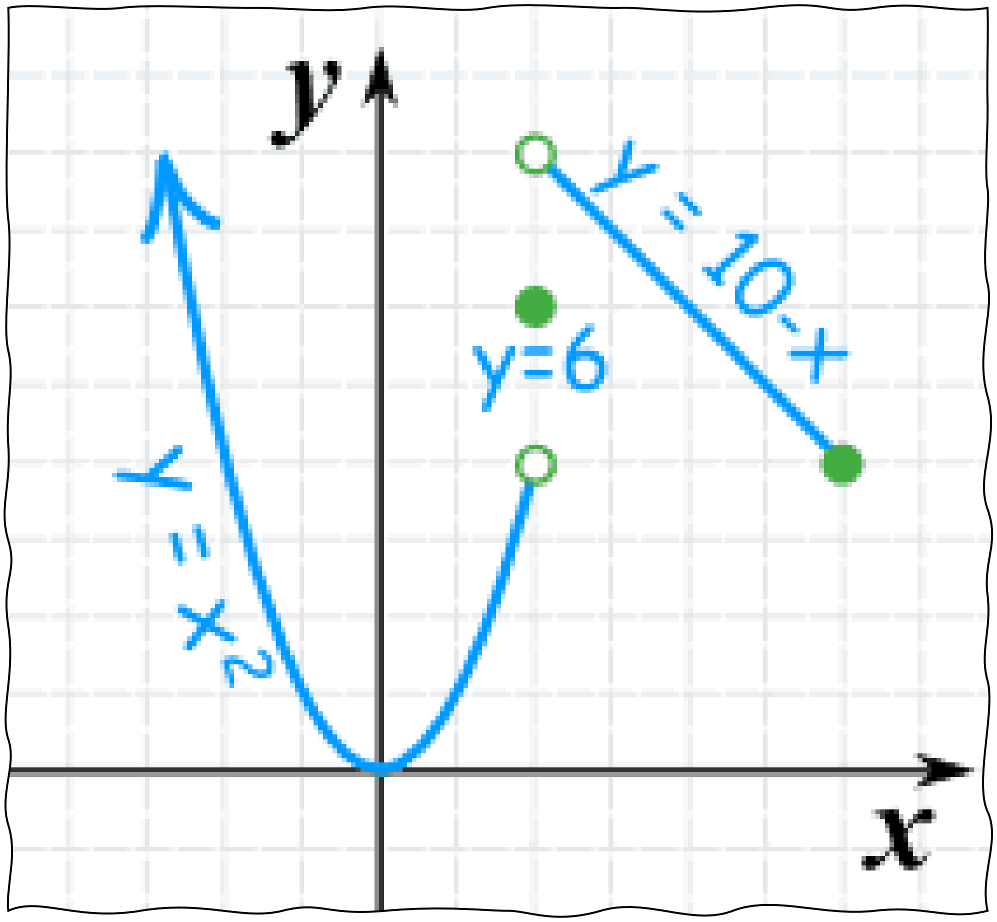
Welcome to AP Calculus AB. The summer work will take a few hours. You'll need a few days to do it.

Below you find Unit 1 material that includes:

1.) Unit 1 HW (blank)
2.) Unit 1 HW (worked out solutions)
3.) Unit 1 Notes (filled in)

The material that you should focus on is the HW section. The notes and solutions are there to help assist your understanding of the material.

WHEN WE START SCHOOL IN THE FALL YOU WILL BE ASSESSED ON THIS CONTENT. We will take the first few days to go over this content as needed.



Name ______________________________

UNIT 1:

LIMITS & CONTINUITY HOMEWORK

1.I

The Concept of Instantaneous Rate of Change

Name ________________________________

Date ____________________ Period ________

1. An object dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ meters in t seconds.

A. How far does the object travel during the time interval [2.5, 3].

B. Find the average velocity of the object over [2.5, 3].

C. Estimate the object's instantaneous velocity at $t = 2.5$ seconds.

Interval	[2.5, 2.51]	[2.5, 2.505]	[2.5, 2.5001]
Average Velocity			

2. Suppose Neil Armstrong decided to throw a golf ball into the air while he was standing on the moon and that the height of the golf ball was modeled by the equation below, where s is measured in feet and t is measured in seconds $s(t) = -2.72t^2 + 26.9t + 6$. Find the best approximation for the instantaneous rate of change (velocity) of the golf ball at 7 seconds.

3. A pendulum swings from the ceiling. Its distance, *d*, in feet, from one wall of the room depends on the number of seconds, *t*, since it was set in motion. Assume that the equation for d as a function of t is $d(t) = 20 + 16\cos\left(\frac{\pi t}{3}\right)$. You want to find out how fast the pendulum is moving at a given instant, *t*, and whether it is approaching or going away from the wall.

A. Find *d* when $t = 4$. What mode should your calculator be set?

B. Estimate the instantaneous rate of change of *d* with respect to t when $t = 2.5$. At that time, is the pendulum approaching the wall or moving away from it? Explain how you know.

4. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Write an algebraic expression to represent the average rate of change of the drug's concentration for the period $t = 0$ hour to $t = 4$ hours after the drug has been administered.

5. Compute $\Delta y/\Delta x$ for the interval $[2,6]$, where $y = 4x - 7$. What is the instantaneous rate of change of y with respect to x at $x = 2$?

6. An initial deposit of \$500 in your bank will have a balance after t years given by the equation $P(t) = 500(1.06)^t$ dollars.

A. What are the units of the rate of change of $P(t)$?

B. Find the average rate of change over $[0, 1]$.

C. Estimate the instantaneous rate of change at $t = 1$ by computing the average rate of change over intervals to the left and right of $t = 1$.

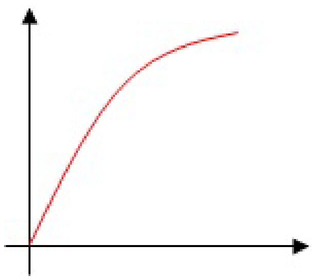
7. Each position graph shown below represents particle motion as a function of time. Match each graph with the proper description:

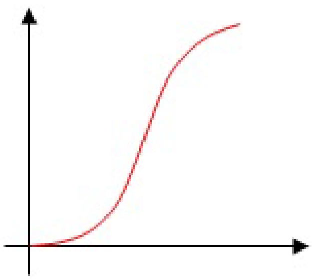
A) Speeding up

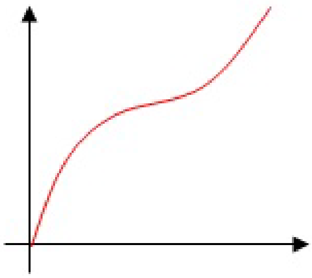
B) Speeding up and then slowing down

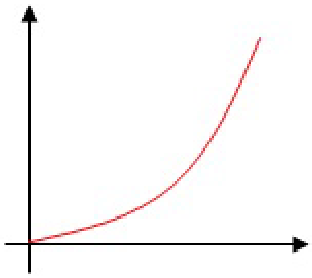
C) Slowing down

D) Slowing down and then speeding up









1.2 Understanding Limits Graphically & Numerically

Name ______________________________

Date ______________________ Period ________

Problems 1 – 4, complete the table and use the result to estimate the limit. Graph the function and verify your result.

1. $\lim\limits_{x \to 2} \frac{x-2}{2x^2-9x+10}$

x	1.99	1.999	2	2.001	2.01
$f(x)$					

2. $\lim\limits_{x \to 0} \frac{\sin x}{x}$

x	-0.01	-0.001	0	0.001	0.01
$f(x)$					

3. $\lim\limits_{x \to 4} \frac{x^2-16}{x-4}$

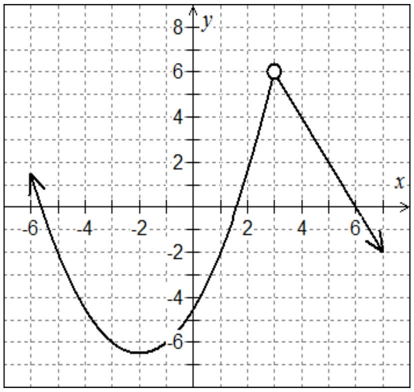
x	3.99	3.999	4	4.001	4.01
$f(x)$					

4. $\lim\limits_{x \to 3} \frac{\sqrt{x+1}-2}{x-3}$

x	2.99	2.999	3	3.001	3.01
$f(x)$					

Problems 5 – 8, use the graph to find the limit, if it exists. If the limit does not exist, explain why.

5.



A. $\lim\limits_{x \to -3} f(x)$

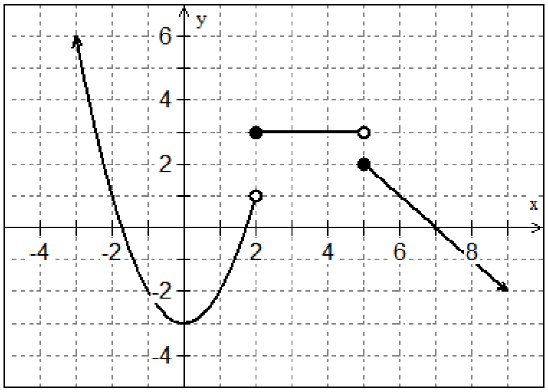
B. $\lim\limits_{x \to -\infty} f(x)$

C. $\lim\limits_{x \to 6} f(x)$

D. $\lim\limits_{x \to 1} f(x)$

E. Does $\lim\limits_{x \to 3} f(x)$ exist? Why or why not?

6.



A. $\lim\limits_{x \to 0} f(x)$

B. $\lim\limits_{x \to 2^-} f(x)$

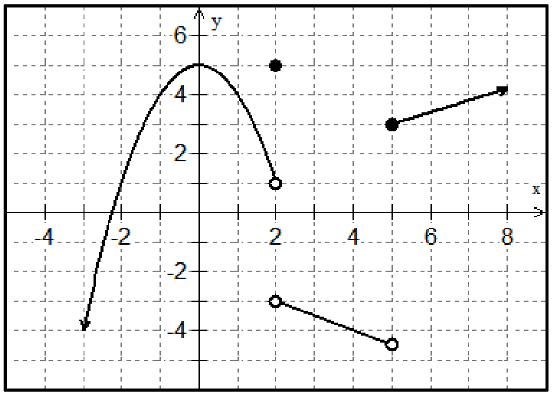
C. $\lim\limits_{x \to 4} f(x)$

D. $\lim\limits_{x \to 5} f(x)$

E. $\lim\limits_{x \to -\infty} f(x)$

F. $\lim\limits_{x \to 7} f(x)$

7.



A. $\lim_{x\to 5^-} f(x)$

B. $\lim_{x\to 2} f(x)$

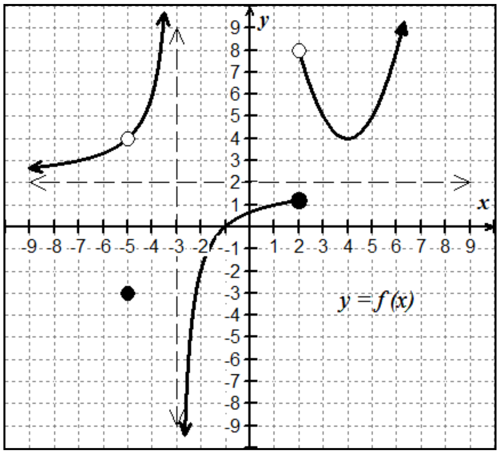
C. $\lim_{x\to 0} f(x)$

D. $\lim_{x\to \infty} f(x)$

E. $\lim_{x\to 4} f(x)$

F. $\lim_{x\to 2^+} f(x)$

8.



A. $\lim_{x\to -3^-} f(x)$

B. $\lim_{x\to -3^+} f(x)$

C. $\lim_{x\to 2^-} f(x)$

D. $\lim_{x\to 2^+} f(x)$

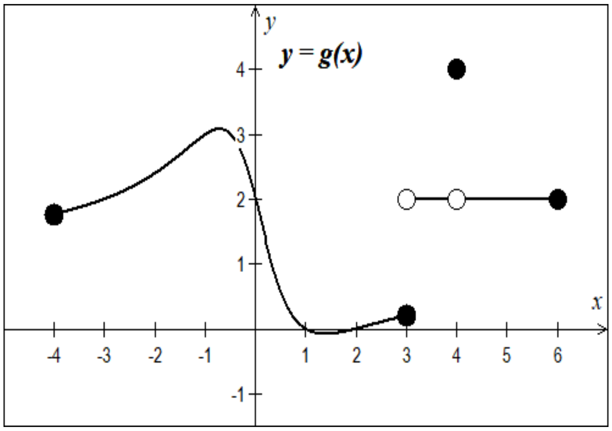
E. $\lim_{x\to \infty} f(x)$

F. $\lim_{x\to -\infty} f(x)$

G. $\lim_{x\to -3} f(x)$

H. $\lim_{x\to 2} f(x)$

Problems 9 – 11, Use the graph of $g(x)$ below to determine if the statements are true or false. If false, explain why.



9. $\lim_{x\to 4} g(x) = 4$

10. $\lim_{x\to 5} g(x) = 2$

11. $\lim_{x\to c} g(x)$ exists for every value of c in the interval $(-4, 6)$

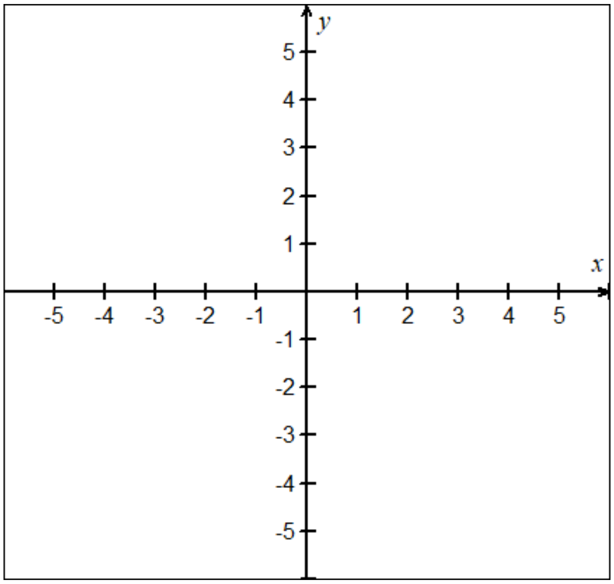
Problems 12 – 13, Sketch a graph of a function that satisfies each of the following conditions.

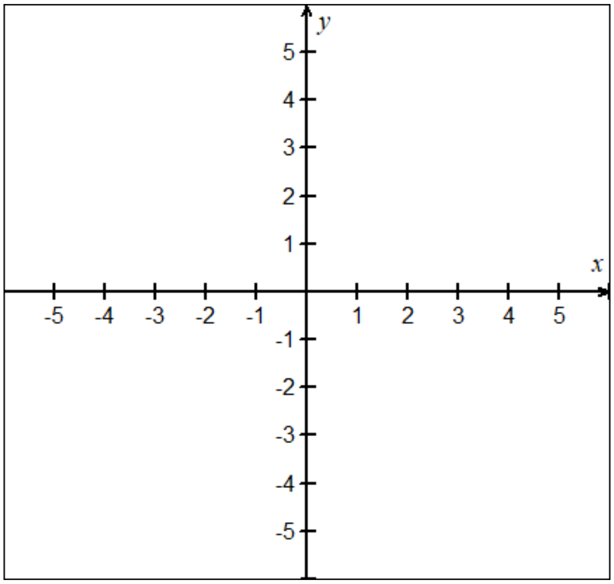
12. $\lim_{x\to 1^-} f(x) = 2$ $\lim_{x\to 1+} f(x) = -3$

$f(1) = 4$

13. $\lim_{x\to -3^-} f(x) = 1$ $\lim_{x\to -3+} f(x) = -4$

$\lim_{x\to 2^-} f(x) = \infty$ $\lim_{x\to 2^+} f(x) = -\infty$





14. Use the table below of the rational function $y = H(x)$ to find the indicated limits. For limits that do not exist, write D.N.E.

x	-1000	-4.001	-4	-3.999	1.999	2	2.001	1000
$H(x)$	-0.998	0.666	undefined	0.666	-4497	undefined	4504	3.0002

A. $\lim_{x\to -4^-} H(x) =$

B. $\lim_{x\to -4^+} H(x) =$

C. $\lim_{x\to -\infty} H(x) =$

D. $\lim_{x\to -4} H(x) =$

E. $\lim_{x\to 2^-} H(x) =$

F. $\lim_{x\to 2+} H(x) =$

G. $\lim_{x\to 2} H(x) =$

H. $\lim_{x\to \infty} H(x) =$

1.3 Properties of Limits

Name ______________________________

Date ____________________ Period ________

Problems 1 – 6, find the limits if $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -3$

1. $\lim_{x\to c}[2f(x) - 3]^3$

2. $\lim_{x\to c}[3f(x) \cdot 2g(x)]$

3. $\lim_{x\to c}\sqrt{2\,f(x) - 4\,g(x)}$

4. $\lim_{x\to c}\frac{5f(x) + 2\,g(x)}{g(x) - f(x)}$

5. $\lim_{x\to c}[f(x) \cdot (g(x) + 5)]$

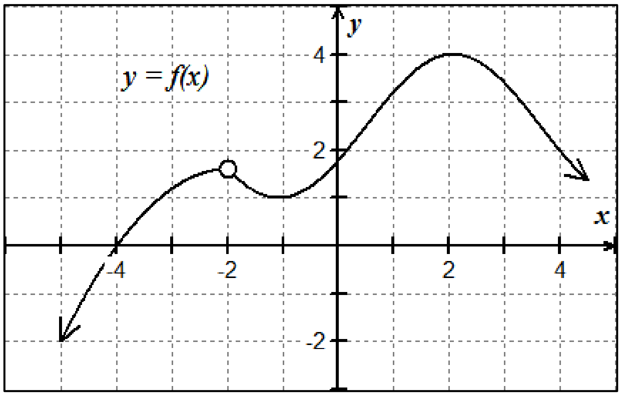
6. $\lim_{x\to c}\frac{[7\,f(x)]^2}{1 - g(x)}$

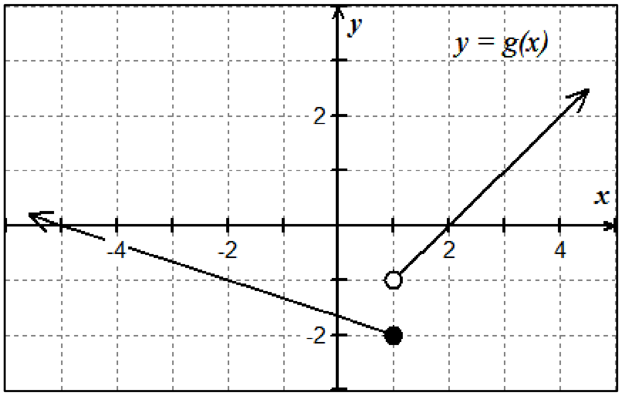
Problems 7 – 8, use the properties of limits to evaluate. Justify each step with a named rule.

7. $\lim_{x\to -3}(5x + 9)$

8. $\lim_{x\to 4}\frac{3x}{x-2}$

Problems 9 – 12, use the graphs of f and g below to evaluate the limits, if they exist.





9. $\lim_{x\to -2}[f(x)+4\,g(x)]$

10. $\lim_{x\to 1}[f(x)\cdot g(x)]$

11. $\lim_{x\to 2}\frac{g(x)}{f(x)}$

12. $\lim_{x\to 4}\left[f\left(g(x)\right)\right]^2$

Problems 13 – 16, True or False.

13. If $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = 0$, then $\lim_{x\to 4}\left[\frac{f(x)}{g(x)}\right]$ does not exist.

14. If $\lim_{x\to 4} f(x) = 0$ and $lim_{x\to 4}\, g(x) = 0$, then $\lim_{x\to 4}\left[\frac{f(x)}{g(x)}\right]$ does not exist.

15. $\lim_{x\to 1}\frac{x^2+4x-5}{x^2+3x-4} = \frac{\lim_{x\to 1}\; x^2+4x-5}{\lim_{x\to 1}\; x^2+3x-4}$

16. $\lim_{x\to 1}\frac{x^2-3}{x^2+5x-4} = \frac{\lim_{x\to 1}\; x^2-3}{\lim_{x\to 1}\; x^2+5x-4}$

Problems 17 – 22, Use the table of values below to find limits for the following:

x	-2	3	6	c
$f(x)$	5	0	1	$\lim_{x\to c} f(x) = 6$
$g(x)$	-3	-2	-4	$\lim_{x\to c} g(x) = -2$
$h(x)$	4	2	3	$\lim_{x\to c} h(x) = 3$

17. $\lim_{x\to c} f\big(g(x)\big)$

18. $\lim_{x\to c} g\big(h(x)\big)$

19. $\lim_{x\to c} \frac{g\left(f(x)\right)}{h\left(g(x)\right)}$

20. $\lim_{x\to c} \frac{f(x)}{f\left(h(x)\right)}$

21. $\lim_{x\to c} f\Big(g\big(h(x)\big)\Big)$

22. $\lim_{x\to c} g\Big(h\big(f(x)\big)\Big)$

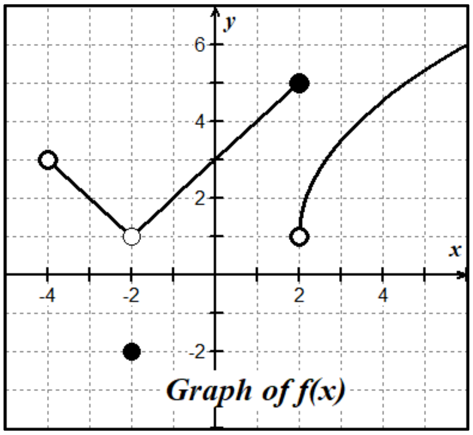
1.4

Limits of Composite Functions

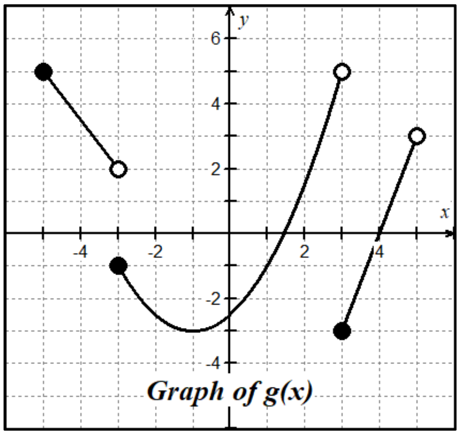
Name ________________________________

Date ____________________ Period ________

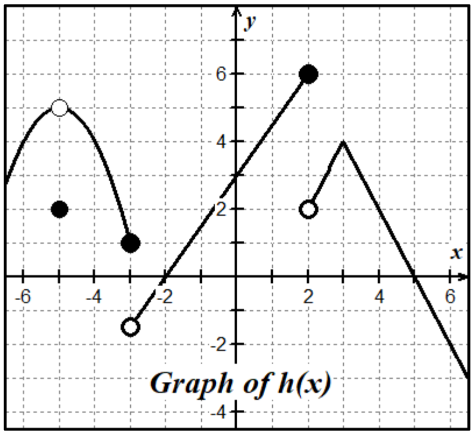
Problems 1 – 8, Use the graphs of $f(x0, g(x)$ and $h(x)$ shown below to answer the following limit questions.



Graph of f(x)



Graph of g(x)



Graph of h(x)

1. $\lim\limits_{x\to -1} [f(x)+g(x)]$

2. $\lim\limits_{x\to 0} [f(x)\cdot h(x)]$

3. $\lim\limits_{x\to -2} f\big(h(x)\big)$

4. $\lim\limits_{x\to 1} \sqrt{f(x)}$

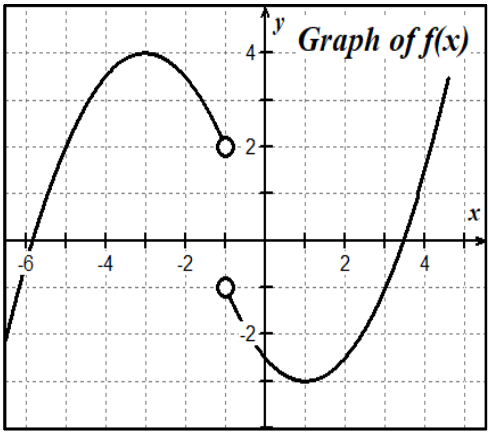
5. $\lim\limits_{x\to 0} h(|x|+3)$

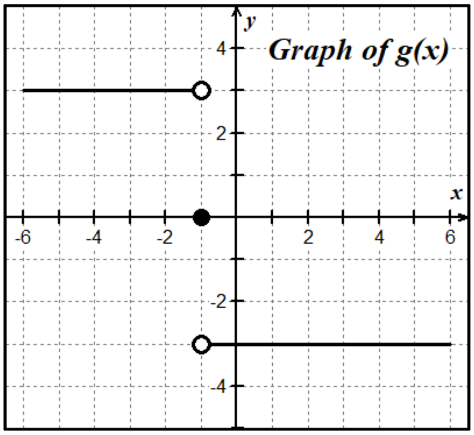
6. $\lim\limits_{x\to -5^{+}} \left[\dfrac{g(x)}{h(x)}\right]$

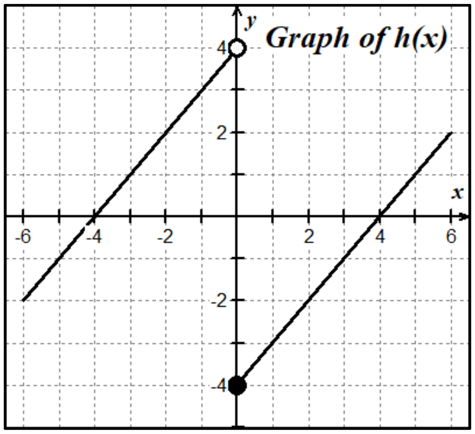
7. $\lim\limits_{x\to -5} g(x+2)$

8. $\lim\limits_{x\to -2} f\sqrt{x+3}$

Problems 9 – 14, use the graphs of f, g and h below to evaluate the limits, if they exist.







9. $\lim\limits_{x\to 5^+} g(25 - x^2)$

10. $\lim\limits_{x\to -1} \left[(g(x))^2 - 5\right]$

11. $\lim\limits_{x\to 0} \dfrac{h(x)}{h(-x)}$

12. $\lim\limits_{x\to 0} h\big(h(x)\big)$

13. $\lim\limits_{x\to -1} f\big(g(x)\big)$

14. $\lim\limits_{x\to -2} f\Big(g\big(h(x)\big)\Big)$

Problems 15 – 16, Let $v(x) = \begin{cases} 3, & x = 0 \\ x, & x \neq 0 \end{cases}$ and $w(x) = \begin{cases} -3, & x < 0 \\ 0, & x = 0 \\ 3, & x > 0 \end{cases}$

15. $\lim\limits_{x\to 0}\big(v(x) \cdot w(x)\big)$

16. $\lim\limits_{x\to 0} \dfrac{w(x)}{v(x)}$

1.5

Finding Limits by Analytic Methods

Name ______________________________

Date ____________________ Period ________

Problems 1 – 16, Find each of the following limits analytically. Show your algebraic analysis.

1. $\lim\limits_{x\to 3}\left(\frac{2}{3}x^2+3x\right)$

2. $\lim\limits_{t\to 4}\frac{t-4}{t^2-16}$

3. $\lim\limits_{x\to -3}\frac{x^2-5x+6}{2x+6}$

4. $\lim\limits_{\theta\to \pi}[\sin^2\theta-3\cos\theta]$

5. $\lim\limits_{x\to 2}\frac{x^3-8}{x-2}$

6. $\lim\limits_{\theta\to \frac{\pi}{3}}\frac{\tan\theta}{\theta^2}$

7. $\lim\limits_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

8. $\lim\limits_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x}$

9. $\lim\limits_{x \to e} \frac{3x}{\ln x}$

10. $\lim\limits_{x \to 2^+} \frac{3x^2+7x+2}{x^2-4}$

11. $\lim\limits_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$

12. $\lim\limits_{x \to \frac{3}{2}} \frac{8x^3-27}{2x-3}$

13. $\lim\limits_{x \to \pi^+} \cot x$

14. $\lim\limits_{x \to 0} \cos(x + \sin x)$

15. $f(x) = \begin{cases} 3x - 1, & x \leq 1 \\ 3x^2, & x > 1 \end{cases}$, find $\lim\limits_{x \to 1} f(x)$

16. $\lim\limits_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

Problems 17 – 32, find the limits, if they exist, for the given piecewise-defined functions.

$$f(x) = \begin{cases} x+3\,, & x < 2 \\ x^2 - 1\,, & 2 \le x < 4 \\ \sqrt{x+5}\,, & x \ge 4 \end{cases}$$

$$g(x) = \begin{cases} 4 - x^2\,, & x \le 1 \\ 7\,, & 1 < x < 3 \\ 2 - x, & x \ge 3 \end{cases}$$

17. $\lim\limits_{x \to 2^+} f(x)$

18. $\lim\limits_{x \to 2^-} f(x)$

19. $\lim\limits_{x \to 2} f(x)$

20. $f(2)$

21. $\lim\limits_{x \to 4^+} f(x)$

22. $\lim\limits_{x \to 4^-} f(x)$

23. $\lim\limits_{x \to 4} f(x)$

24. $f(4)$

25. $\lim\limits_{x \to 1^-} g(x)$

26. $\lim\limits_{x \to 1^+} g(x)$

27. $\lim\limits_{x \to 1} g(x)$

28. $g(1)$

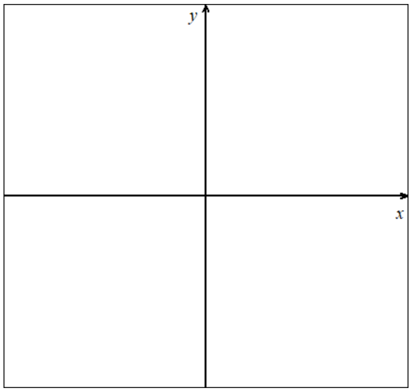
29. $\lim\limits_{x \to 3^-} g(x)$

30. $\lim\limits_{x \to 3^+} g(x)$

31. $\lim\limits_{x \to 3} g(x)$

32. $g(3)$

33. If $2 \le f(x) \le x^2 + 2$ for all *x*, find $\lim\limits_{x \to 0} f(x)$. Sketch a graph to illustrate .



34. If $\lim\limits_{x \to c} f(x) = -5$ and $\lim\limits_{x \to c} g(x) = 8$, find $\lim\limits_{x \to c} \dfrac{2\,f(x)}{g(x) - f(x)}$

1.6

Limits of Transcendental Functions

Name ______________________________

Date ____________________ Period ________

Problems 1 – 6, find the limits of each exponential function. If necessary, sketch a graph of the function to help you determine the behavior and limit.

1. $\lim_{x \to -2} 3^{x+4} - 2$

2. $\lim_{x \to \infty} 2^{x-3} + 4$

3. $\lim_{x \to -\infty} -(0.25)^{-x+1} + 5$

4. $\lim_{x \to 3} \left(\frac{1}{2}\right)^{3-x} - 1$

5. $\lim_{x \to -1} -4^{-x-1} + 5$

6. $\lim_{x \to \infty} -\left(\frac{1}{3}\right)^{2-x} - 3$

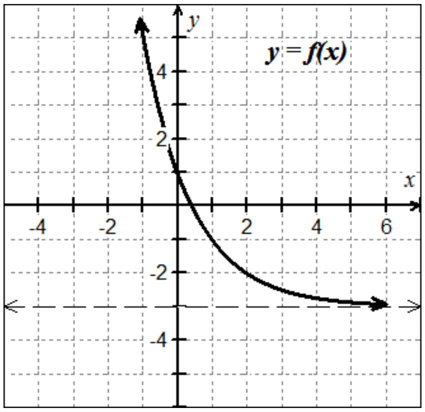
Problems 7 – 10, use the graph of $y = f(x)$ at right, to find each of the following limits.

7. $\lim_{x \to \infty} f(x) =$ _________

8. $\lim_{x \to -\infty} f(x) =$ _________

9. $\lim_{x \to 0} f(x) =$ _________

10. $\lim_{x \to 2} f(x) =$ _________



Problems 11 – 28, evaluate the limits, if they exist.

11. $\lim_{x\to 0} \frac{\sin\ 3x}{x}$

12. $\lim_{x\to 0} \frac{\sin x \sec x}{x}$

13. $\lim_{x\to 0} \frac{e^x\ \cos x}{2}$

14. $\lim_{x\to \frac{\pi}{2}} \frac{1\ -\ \cos x}{3x}$

15. $\lim_{x\to 0} \frac{x\ +\ \sin x}{x}$

16. $\lim_{x\to 4}\ e^{3-x} + 4$

17. $\lim_{\theta\to \frac{\pi}{3}} \frac{\cos^2\theta}{1\ -\ \sin\theta}$

18. $\lim_{x\to 0} \frac{\sin x}{3x^2 - x}$

19. $\lim_{x\to 2} \begin{cases} 3x^2 - 2x\,, & x < 2 \\ 7 - \cos\left(\frac{\pi x}{3}\right), & x \geq 2 \end{cases}$

20. $\lim_{x\to 3} \begin{cases} \ln x\,, & 0 < x < 2 \\ x^2 \ln x\,, & 2 \leq x \leq 4 \end{cases}$

21. $\lim_{x\to\frac{\pi}{3}} \frac{2\sin^2 x\ + 3\sin x\ - 2}{2\sin x\ - 1}$

22. $\lim_{x\to 0} \frac{5x +\ \sin 2x}{x}$

23. $\lim_{x\to 0} \frac{3^x \cos x}{4}$

24. $\lim_{x\to -3} \left(\frac{1}{4}\right)^{-x-4} + 5$

25. $\lim_{\theta\to 0} \frac{5\cos\theta\ -\ 5}{\theta}$

26. $\lim_{x\to -3} \frac{(x+3)\ln(x+5)}{x^2\ - 9}$

27. $\lim_{\theta \to 0} \frac{\sin 3\theta \ + \ 1 \ - \ \cos\theta}{\theta}$

28. $\lim_{\theta \to 0} \frac{\cos\theta \sin\theta - \sin\theta}{\theta^2}$

Problems 29 – 30, Use the Squeeze Theorem to evaluate the following limits.

29. $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

30. If $0 \leq f(x) \leq c$ for some real number c, prove that $\lim_{x \to 0} x^2 f(x) = 0$

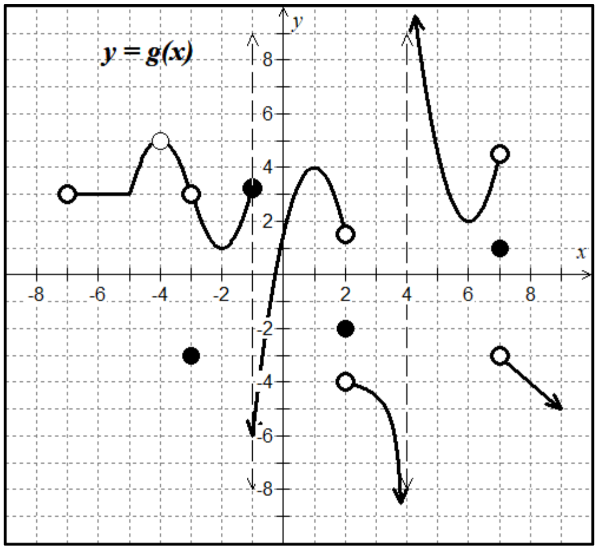
1.7 Limits and Continuity

Name ______________________________

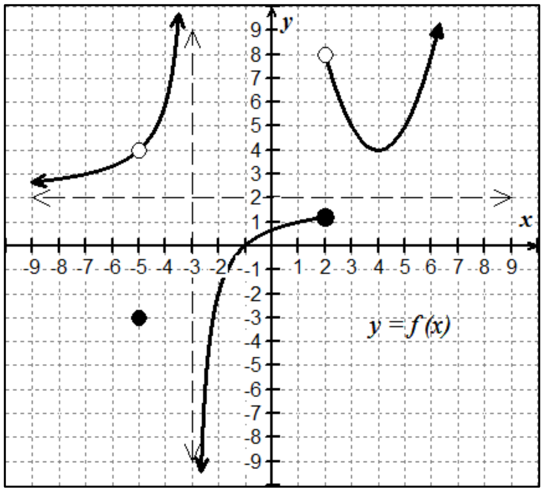
Date ____________________ Period ________

1. Use the graph of the function $y = g(x)$ shown below, to evaluate each of the following.

A. $\lim_{x\to -3} g(x) =$	**B.** $\lim_{x\to -1^-} g(x) =$
C. $\lim_{x\to -1^+} g(x) =$	**D.** $g(-1) =$
E. $\lim_{x\to -1} g(x) =$	**F.** $\lim_{x\to 4^+} g(x) =$
G. $\lim_{x\to 4^-} g(x) =$	**H.** $\lim_{x\to 7} g(x) =$
J. $g(7) =$	**K.** $\lim_{x\to 2^+} g(x) =$



2. Use the graph of the function $y = f(x)$ shown below, to evaluate each of the following.



A. On the interval $x \in (-\infty, \infty)$, list the largest intervals for which $f(x)$ is continuous.

B. Find the smallest value k, such that the function is continuous on (k, ∞)

C. Find the smallest value k, such that the function is continuous on $[k, \infty)$

D. Find the largest value of b such that $y = f(x)$ is continuous on $(-3, b]$ but not continuous in $(-3, b + 1]$. State all values of b that would work.

Problem 3 – 6, determine the points, classify the type for each as removable, non-removable, jump, or infinite.

3. $f(x) = \dfrac{1}{(x-3)^2}$

4. $g(x) = \dfrac{x-4}{x^2-9x+20}$

5. $h(x) = \dfrac{|x+2|}{x+2}$

6. $f(x) = \begin{cases} x+1 & x<2 \\ -1 & x=2 \\ x^2+1 & x>2 \end{cases}$

Problems 7 – 8, use the three-part definition of continuity to determine if the given functions are continuous at the indicated values of *x*.

7. $f(x) = \begin{cases} e^x \cos x\ , & x \geq \pi \\ e^x \tan\left(\frac{3x}{4}\right), & x < \pi \end{cases}$ at $x = \pi$

8. $g(x) = \begin{cases} \frac{x^2-9}{x+3} & x \neq -3 \\ 5 & x = -3 \end{cases}$ at $x = -3$

Problems 9 – 12, find all value(s) of *a, b, c* or *k* that make the function continuous everywhere.

9. $f(x) = \begin{cases} kx^2 & x \leq 3 \\ 4x - 11 & x > 3 \end{cases}$

10. $g(x) = \begin{cases} cx^2 & x < 1 \\ 4 & x = 1 \\ -x^3 + kx & x > 1 \end{cases}$

11. $h(x) = \begin{cases} \pi & x < 0 \\ x^2 + ax + b & 0 \leq x \leq 1 \\ 6x + 5 & x > 1 \end{cases}$

12. $f(x) = \begin{cases} x^2 & x < 1 \\ \sin(bx) & x \geq 1 \end{cases}$

13. Consider the function $y = f(x)$ to answer the following. $f(x) = \begin{cases} -3 & x \leq -1 \\ mx + k & -1 < x < 4 \\ 3 & x \geq 4 \end{cases}$

A. What two limits must be equal in order for the function to be continuous at $x = -1$?

B. What two limits must be equal in order for the function to be continuous at $x = 4$?

C. Find the values of m and k so that the function is continuous everywhere.

14. If $y = f(x)$ is continuous for all $x \neq \frac{1}{2}$, evaluate the following. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{2x^2 + 3x - 2}, & x \neq -2 \\ k\ , & x = -2 \end{cases}$

A. $\lim_{x \to \frac{1}{2}^+} f(x) =$

B. $\lim_{x \to 1} f(x) =$

C. What is the value of k ?

1.8

Infinite Limits and Limits at Infinity

Name ______________________________

Date ____________________ Period ________

Problem 1 – 4, Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$, then identify all horizontal or slant asymptotes.

1. $f(x) = \dfrac{7 + 2x - 5x^2}{2x^2 - 7x - 4}$

2. $f(x) = \dfrac{2x - 3}{\sqrt{4x^2 + 3}}$

3. $f(x) = \dfrac{3 + 4x - 2x^3}{x^2 + 1}$

4. $f(x) = \dfrac{x^{4/3} + x^{1/3}}{\left(4x^{2/3} + 1\right)^2}$

Problems 5 – 6, sketch a function that satisfies the stated conditions. Include asymptotes.

5.

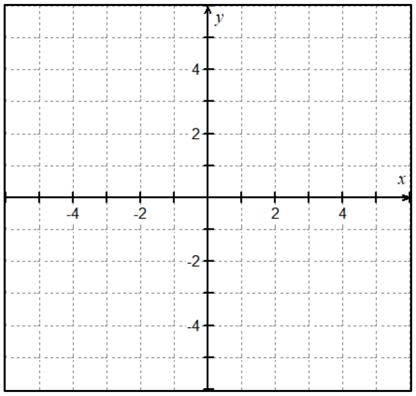
$\lim\limits_{x\to 3} f(x) = -2$

$\lim\limits_{x\to -2^-} f(x) = -\infty$

$\lim\limits_{x\to -2^+} f(x) = +\infty$

$\lim\limits_{x\to -\infty} f(x) = 3$

$\lim\limits_{x\to +\infty} f(x) = \infty$



6.

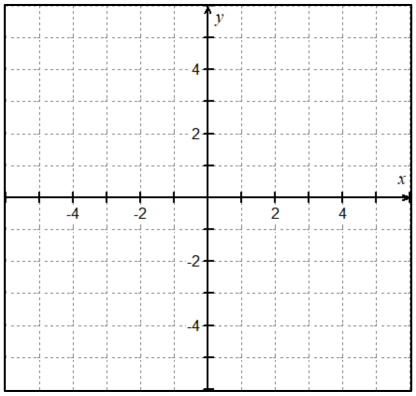
$\lim\limits_{x\to -3^-} f(x) = +\infty$

$\lim\limits_{x\to -3^+} f(x) = +\infty$

$\lim\limits_{x\to 2} f(x) = 3$

$\lim\limits_{x\to -\infty} f(x) = -3$

$\lim\limits_{x\to +\infty} f(x) = -3$



Problems 7 – 10, Identify all vertical asymptotes and find $\lim_{x \to a^+} f(x)$; $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a} f(x)$, where a is the x-value of the asymptote.

7. $f(x) = \dfrac{x+4}{x^2 + 9x + 20}$

8. $f(x) = \dfrac{2x^2 - x - 15}{x^2 - 5x + 6}$

9. $f(x) = \dfrac{\ln(x^2+1)}{x+1}$

10. $f(x) = \dfrac{x^2 + 3x - 18}{x^2 - 6x + 9}$

Problems 11 – 12, For the piecewise functions, find the limit as $x \to \infty$, $x \to -\infty$, $x \to 0^-$, and $x \to 0^+$.

11. $f(x) = \begin{cases} \dfrac{2x+3}{x-1}, & x < 0 \\ \dfrac{1}{x}, & x \geq 0 \end{cases}$

12. $g(x) = \begin{cases} \dfrac{1}{x^2}, & x < 0 \\ \dfrac{2x}{x+1}, & x \geq 0 \end{cases}$

Problems 13 – 20, Evaluate the following limits without the aid of a calculator.

13. $\lim_{x \to \infty} \left[\left(\frac{3x^2-1}{x^2} \right) \left(\frac{2}{x} - 1 \right) \right]$

14. $\lim_{x \to -\infty} \left(\frac{5+4x-3x^2}{2x^2+1} \right)$

15. $\lim_{x \to \infty} (e^{-x} \sin x)$

16. $\lim_{x \to 3^+} \ln(x^2 - 9)$

17. $\lim_{x \to -1^-} \frac{x+1}{x^4-1}$

18. $\lim_{x \to 0^-} \frac{2^x}{x}$

19. $\lim_{x \to \infty} \frac{x^2 + 3x}{\sqrt{2x^2 + 1}}$

20. $\lim_{x \to \infty} \frac{4x + 9}{2x^2 - x + 6}$

1.q

Intermediate Value Theorem

Name ____

Date __________________________ Period ________

Problems 1 – 13, Use the Intermediate Value Theorem to complete.

1. In the function $f(x) = x^3 - x - 1$, it can be shown that $f(1) = -1$ and $f(2) = 5$. Complete the table below to find an approximation for a solution of the interval $[1, 2]$.

$\boldsymbol{x}$	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\boldsymbol{f(x)}$											

2. Find the value of c guaranteed by the Intermediate Value Theorem. $f(x) = x^2 + 4x - 13$ on $[0, 4]$ such that $f(c) = 8$

3. Show that $g(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval $[-1, 2]$.

4. Between which of the following two values does the equation $3x^3 + 5x - 11 = 0$ have a solution?

(A) $[-2, -1]$ **(B)** $[0, 1]$ **(C)** $[-1, 0]$ **(D)** $[1, 2]$

5. Given the function $f(x) = \frac{2x-3}{2x-5}$, determine which interval(s) satisfies the conditions for the Intermediate Value Theorem, such that $f(x) = 0$.

(A) One solution between x = 0 and x = 1

(B) One solution between $x = 1$ and $x = 2$

(C) One solution between $x = 1$ and $x = 2$ and one solution between $x = 2$ and $x = 3$

(D) One solution between $x = 2$ and $x = 3$

---

6. Apply the Intermediate Value Theorem, if possible, on [1, 2] so that $f(c) = 9$ for the function $f(x) = x^3 + x$.

---

7. A delivery van travels along a straight road. During the time interval $0 \leq t \leq 30$ seconds, the van's velocity in feet per second is a continuous function. Use the table below to find the minimum number of times that the van must have been stopped. Justify your answer.

***t* (sec)**	0	5	7	12	18	22	30
$V(t)$ (ft/sec)	-28	-60	-15	8	24	-4	10

---

8. Explain why the Intermediate Value Theorem does not apply for guaranteeing that a zero exists for the function $f(x) = x^2 + 2x + 5$ over [0, 6].

9. The functions f and g are continuous. The function h is given by $h(x) = f(g(x)) - x$. The table below gives values of the functions. Explain why there must be a value c for $1 < c < 5$ such that $h(c) = -2$.

x	1	2	3	4	5
$f(x)$	0	9	7	-3	8
$g(x)$	4	6	-4	1	3

---

10. Given $f(x) = \frac{x}{x-3}$ on the interval $[-2, 2]$. Determine if the IVT applies. State why or why not. Then, find the value of c such that $f(c) = \frac{1}{3}$.

---

11. Show that there is a value c with $0 < c < 2$ such that $x^2 + \cos \pi x = 4$. Then, use a graphing utility to find the approximate value of c.

12. Does the IVT apply to the function $h(x) = \frac{x^2 + x}{x - 2}$ on the interval $[2.5,\ 5]$? If so, find the value of c guaranteed to exist, such that $h(c) = 12$.

---

13. Does the IVT apply to the function $f(x) = -\left(\frac{1}{2}\right)^{3-x} - 3$ on the interval $[2,5]$ for $f(c) = -4$?

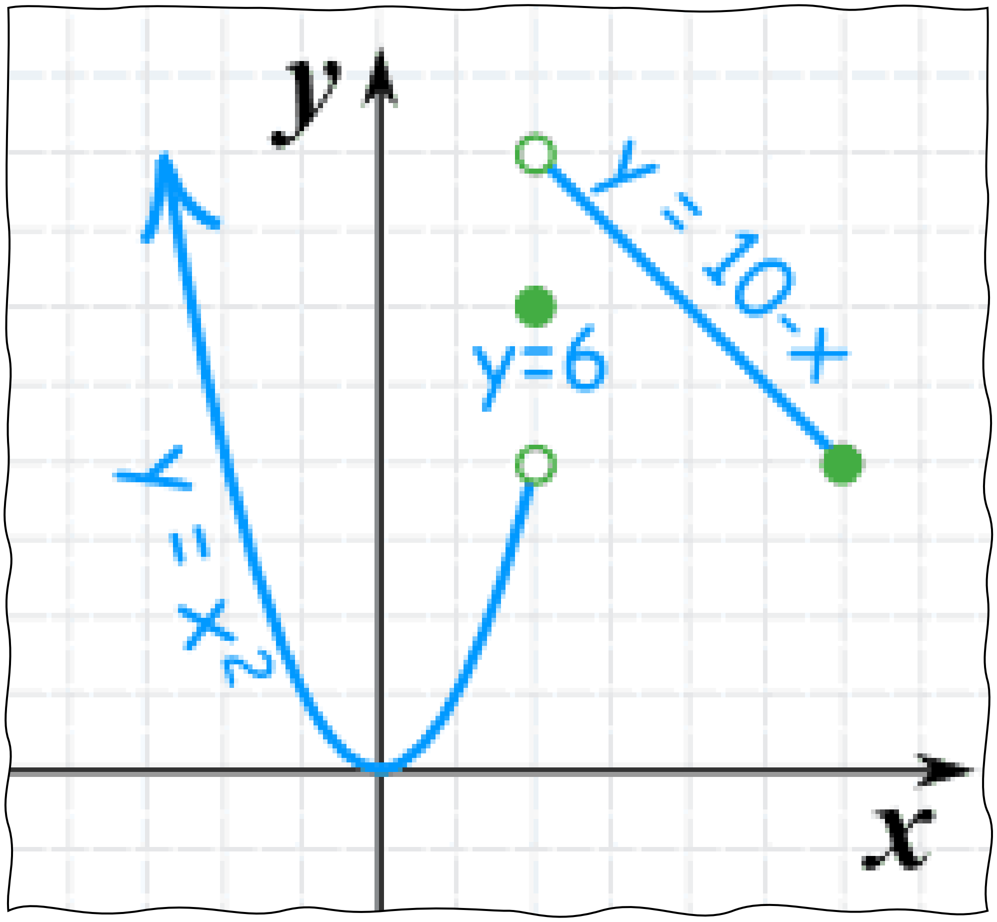
________ **14.** Let f be a continuous function on the closed interval $[-2, 7]$. If $f(-2) = -3$ and $f(7) = 4$, then the Intermediate Value Theorem guarantees that

(A) $f(0) < 0$

(B) $-3 \leq f(x) \leq 4$ for all x between -2 and 7.

(C) $f(c) = 1$ for at least one c between -2 and 7

(D) $f(c) = 0$ for at least one c between -3 and 4



Name ______________________________________

UNIT 1:

LIMITS & CONTINUITY HOMEWORK

1.1

The Concept of Instantaneous Rate of Change

Name ____________________________

Date __________________ Period ________

1. An object dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ meters in t seconds.

A. How far does the object travel during the time interval [2.5, 3].

$\Delta s = S(3) - S(2.5) \rightarrow 44.1 - 30.625$

$\Delta S = 13.475m$

B. Find the average velocity of the object over [2.5, 3].

$AROC = \frac{S(3) - S(2.5)}{3 - 2.5} = \frac{13.475}{0.5} \frac{m}{s}$

26.95 m/s

C. Estimate the object's instantaneous velocity at $t = 2.5$ seconds.

Interval	[2.5, 2.51]	[2.5, 2.505]	[2.5, 2.5001]
Average Velocity	24.549	24.5245	24.500

IROC @ t = 2.5s is ≈ 24.500 m/s

2. Suppose Neil Armstrong decided to throw a golf ball into the air while he was standing on the moon and that the height of the golf ball was modeled by the equation below, where s is measured in feet and t is measured in seconds $s(t) = -2.72t^2 + 26.9t + 6$. Find the best approximation for the instantaneous rate of change (velocity) of the golf ball at 7 seconds.

$IROC = \frac{S(7.0001) - S(7)}{7.0001 - 7} = \frac{61.01888 - 61.02}{0.0001} \approx -11.180 \frac{ft}{sec}$

3. A pendulum swings from the ceiling. Its distance, d, in feet, from one wall of the room depends on the number of seconds, t, since it was set in motion. Assume that the equation for d as a function of t is $d(t) = 20 + 16\cos\left(\frac{\pi t}{3}\right)$. You want to find out how fast the pendulum is moving at a given instant, t, and whether it is approaching or going away from the wall.

A. Find d when $t = 4$. What mode should your calculator be set?

d(4) = 12 ft, radian mode

B. Estimate the instantaneous rate of change of d with respect to t when $t = 2.5$. At that time, is the pendulum approaching the wall or moving away from it? Explain how you know.

$Rate = \frac{d(2.5001) - d(2.5)}{2.5001 - 2.5} \approx -8.00$ ft/sec

Pendulum is approaching the wall. The rate is negative, so the distance is decreasing.

4. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Write an algebraic expression to represent the average rate of change of the drug's concentration for the period $t = 0$ hour to $t = 4$ hours after the drug has been administered.

$$\frac{\Delta C}{\Delta t} = \frac{C(4)-C(0)}{4-0} = \underline{\underline{\frac{C(4)-C(0)}{4}}}$$

5. Compute $\Delta y/\Delta x$ for the interval $[2,6]$, where $y = 4x - 7$. What is the instantaneous rate of change of y with respect to x at $x = 2$?

$$\frac{\Delta y}{\Delta x} = \frac{f(6)-f(2)}{6-2} = \frac{17-1}{4} = \underline{\underline{4}} \qquad \frac{f(2.001)-f(2)}{2.001-2} = \underline{\underline{4}}$$

AROC = 4 = IROC

6. An initial deposit of \$500 in your bank will have a balance after t years given by the equation $P(t) = 500(1.06)^t$ dollars.

A. What are the units of the rate of change of $P(t)$? dollars per year

B. Find the average rate of change over $[0, 1]$.

$$\frac{P(1)-P(0)}{1-0} = \frac{530-500}{1} = \$\underline{\underline{30}}$$

C. Estimate the instantaneous rate of change at $t = 1$ by computing the average rate of change over intervals to the left and right of $t = 1$.

$$\frac{P(1)-P(0.999)}{1-0.999} = \$30.88 \qquad \frac{P(1.001)-P(1)}{1.001-1} = \$30.88$$

<u>about \$30.88 per year</u>

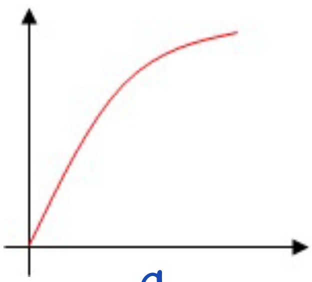
7. Each position graph shown below represents particle motion as a function of time. Match each graph with the proper description:

A) Speeding up

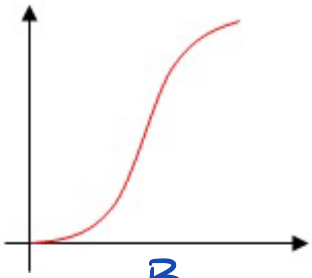
B) Speeding up and then slowing down

C) Slowing down

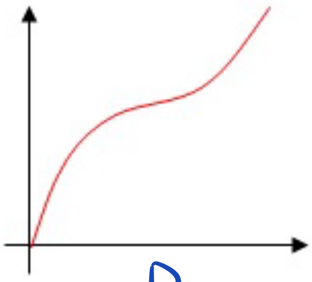
D) Slowing down and then speeding up



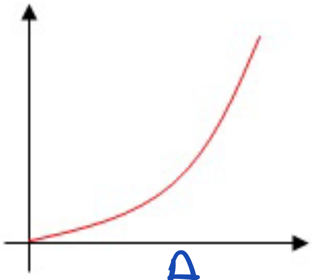
C



B



D



A

1.2 Understanding Limits Graphically & Numerically

Name ______________________________

Date ____________________ Period ________

Problems 1 – 4, complete the table and use the result to estimate the limit. Graph the function and verify your result.

1. $\lim_{x\to 2} \frac{x-2}{2x^2-9x+10} = -1.000$

x	1.99	1.999	2	2.001	2.01
$f(x)$	-.980	-.998	dne	-1.002	-1.020

2. $\lim_{x\to 0} \frac{\sin x}{x} = 1.000$

x	-0.01	-0.001	0	0.001	0.01
$f(x)$	.999	.999	dne	.999	.999

3. $\lim_{x\to 4} \frac{x^2-16}{x-4} = 8.000$

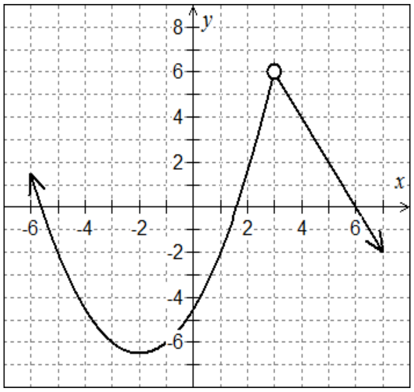
x	3.99	3.999	4	4.001	4.01
$f(x)$	7.99	7.999	dne	8.001	8.01

4. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = 0.250$

x	2.99	2.999	3	3.001	3.01
$f(x)$	.250	.250	dne	.249	.249

Problems 5 – 8, use the graph to find the limit, if it exists. If the limit does not exist, explain why.

5.



A. $\lim_{x\to -3} f(x)$

-6

B. $\lim_{x\to -\infty} f(x)$

∞

C. $\lim_{x\to 6} f(x)$

0

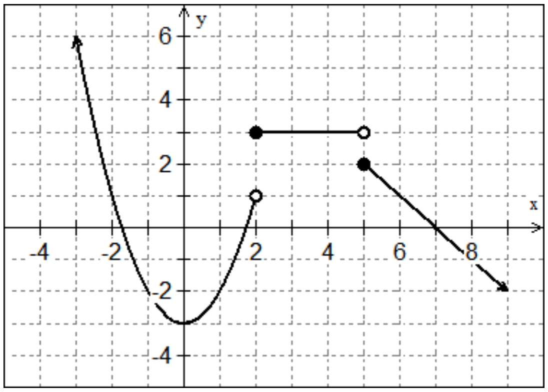
D. $\lim_{x\to 1} f(x)$

-2

E. Does $\lim_{x\to 3} f(x)$ exist? Why or why not? yes

$\lim_{x\to 3^+} f(x) = 6 = \lim_{x\to 3^-} f(x)$

6.



A. $\lim_{x\to 0} f(x)$ -3

B. $\lim_{x\to 2^-} f(x)$

1

C. $\lim_{x\to 4} f(x)$

3

D. $\lim_{x\to 5} f(x)$ dne

$\lim_{x\to 5^+} f(x) \neq \lim_{x\to 5^-} f(x)$

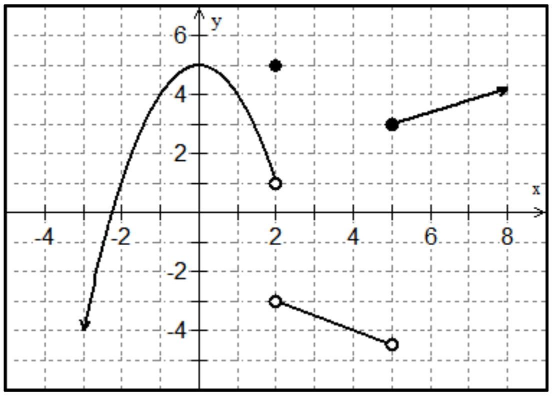
E. $\lim_{x\to -\infty} f(x)$

∞

F. $\lim_{x\to 7} f(x)$

0

7.



A. $\lim_{x\to5^-} f(x)$

-4.5

B. $\lim_{x\to2} f(x)$ dne

$\lim_{x\to2^+} f(x) \neq \lim_{x\to2^-} f(x)$

C. $\lim_{x\to0} f(x)$

5

D. $\lim_{x\to\infty} f(x)$

∞

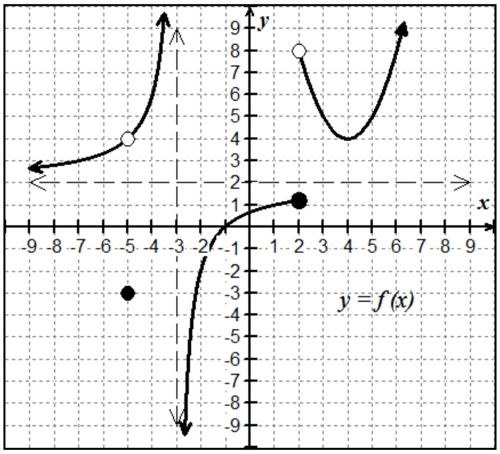
E. $\lim_{x\to4} f(x)$

-4

F. $\lim_{x\to2^+} f(x)$

-3

8.



A. $\lim_{x\to-3^-} f(x)$

∞

B. $\lim_{x\to-3^+} f(x)$

$-\infty$

C. $\lim_{x\to2^-} f(x)$

1

D. $\lim_{x\to2^+} f(x)$

8

E. $\lim_{x\to\infty} f(x)$

∞

F. $\lim_{x\to-\infty} f(x)$

2

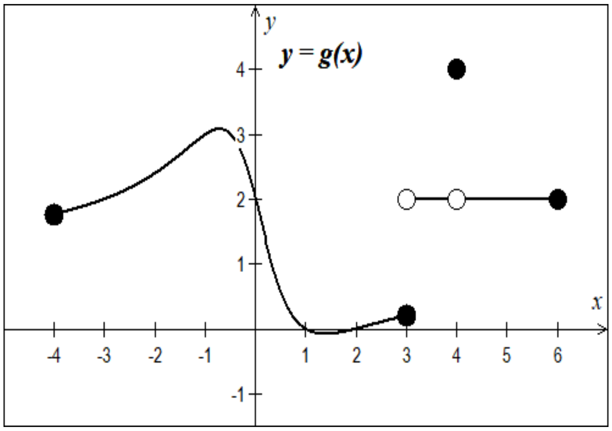
G. $\lim_{x\to-3} f(x)$ dne

$\lim_{x\to^-3^-} f(x) \neq \lim_{x\to^-3^+} f(x)$

H. $\lim_{x\to2} f(x)$ dne

$\lim_{x\to2^-} f(x) \neq \lim_{x\to2^+} f(x$

Problems 9 – 11, Use the graph of $g(x)$ below to determine if the statements are true or false. If false, explain why.



9. $\lim_{x\to4} g(x) = 4$

False $\lim_{x\to4} g(x) = 2$ but $g(4) = 4$

10. $\lim_{x\to5} g(x) = 2$

True

11. $\lim_{x\to c} g(x)$ exists for every value of c in the interval $(-4, 6)$

False $\lim_{x\to3} g(x)$ d.n.e.

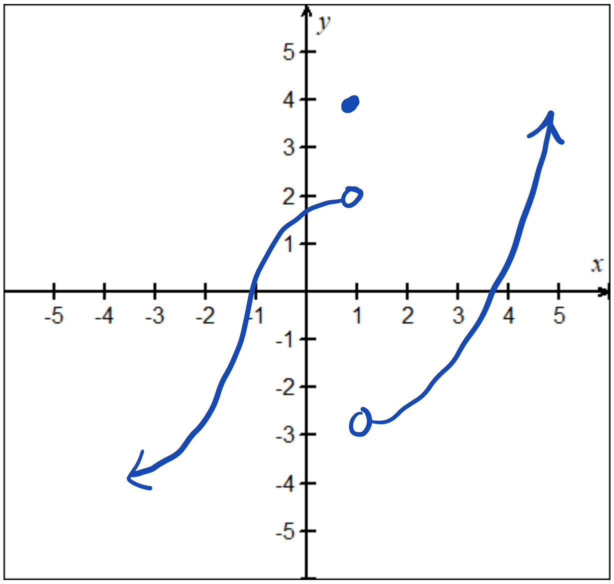
Problems 12 – 13, Sketch a graph of a function that satisfies each of the following conditions.

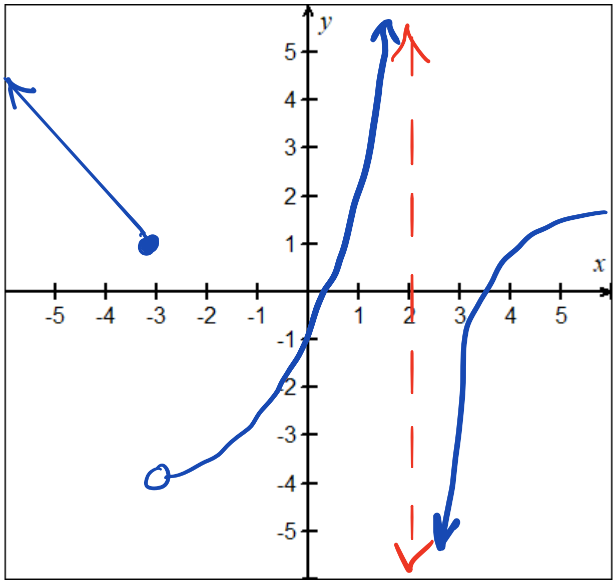
12. $\lim_{x\to 1^-} f(x) = 2$ $\lim_{x\to 1+} f(x) = -3$

$f(1) = 4$

13. $\lim_{x\to -3^-} f(x) = 1$ $\lim_{x\to -3+} f(x) = -4$

$\lim_{x\to 2^-} f(x) = \infty$ $\lim_{x\to 2^+} f(x) = -\infty$





Answers may vary

14. **Use the table below of the rational function $y = H(x)$ to find the indicated limits. For limits that do not exist, write D.N.E.**

x	-1000	-4.001	-4	-3.999	1.999	2	2.001	1000
$H(x)$	-0.998	0.666	undefined	0.666	-4497	undefined	4504	3.0002

A. $\lim_{x\to -4^-} H(x) = \frac{2}{3}$

B. $\lim_{x\to -4^+} H(x) = \frac{2}{3}$

C. $\lim_{x\to -\infty} H(x) = -1$

D. $\lim_{x\to -4} H(x) = \frac{2}{3}$

E. $\lim_{x\to 2^-} H(x) = -\infty$

F. $\lim_{x\to 2+} H(x) = \infty$

G. $\lim_{x\to 2} H(x) =$ D.N.E.

H. $\lim_{x\to \infty} H(x) = 3$

1.3

Properties of Limits

Name ______________________________

Date ____________________ Period ________

Problems 1 – 6, find the limits if $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -3$

1. $\lim_{x\to c}[2f(x) - 3]^3$

$$\left[2\cdot \lim_{x\to c} f(x) - \lim_{x\to c} 3\right]^3$$

$$[2(4)-3]^3 = \underline{\underline{125}}$$

2. $\lim_{x\to c}[3f(x) \cdot 2g(x)]$

$$3\lim_{x\to c} f(x)\cdot 2\lim_{x\to c} g(x)$$

$$3(4)\cdot 2(-3) = \underline{\underline{-72}}$$

3. $\lim_{x\to c}\sqrt{2f(x) - 4g(x)}$

$$\sqrt{2\lim_{x\to c} f(x) - 4\lim_{x\to c} g(x)}$$

$$\sqrt{2(4)-4(-3)} = \underline{\underline{2\sqrt{5}}}$$

4. $\lim_{x\to c}\frac{5f(x)+2g(x)}{g(x)-f(x)}$

$$\frac{5(4)+2(-3)}{-3-4}$$

$$= \frac{20-6}{-7} = \frac{14}{-7} = \underline{\underline{-2}}$$

5. $\lim_{x\to c}[f(x)\cdot (g(x)+5)]$

$$\lim_{x\to c} f(x)\cdot \lim_{x\to c}[g(x)+5]$$

$$4(-3+5) = \underline{\underline{8}}$$

6. $\lim_{x\to c}\frac{[7f(x)]^2}{1-g(x)}$

$$\frac{\left[7\lim_{x\to c} f(x)\right]^2}{\lim_{x\to c} 1 - \lim_{x\to c} g(x)} = \frac{49(4)^2}{1-(-3)} = \underline{\underline{196}}$$

Problems 7 – 8, use the properties of limits to evaluate. Justify each step with a named rule.

7. $\lim_{x\to -3}(5x+9)$

$$\lim_{x\to -3} 5x + \lim_{x\to -3} 9$$ Sum Rule

$$= -15 + 9$$ Substitution

$$= \underline{\underline{-6}}$$

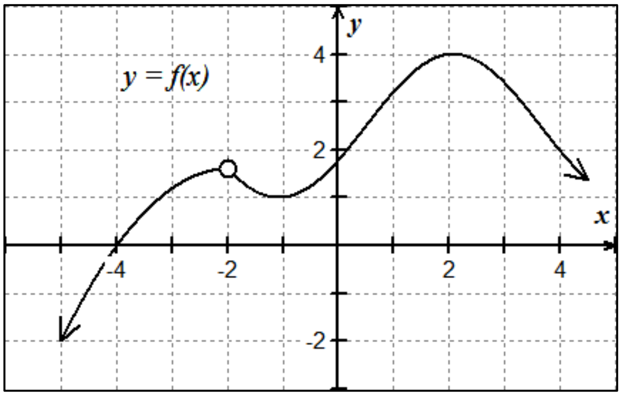
8. $\lim_{x\to 4}\frac{3x}{x-2}$

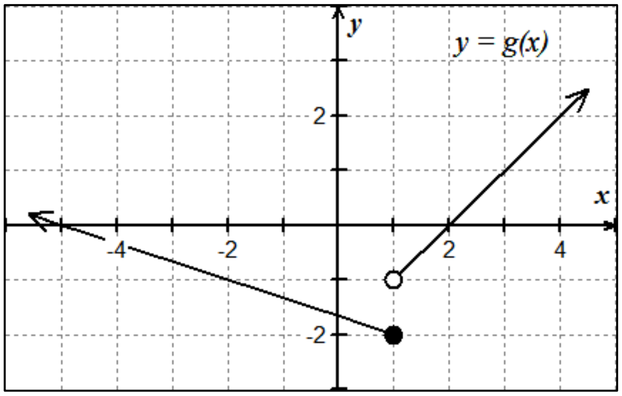
$$\frac{\lim_{x\to 4} 3x}{\lim_{x\to 4}(x-2)}$$ Quotient Rule

$$= \frac{12}{2}$$ Substitution

$$= \underline{\underline{6}}$$

Problems 9 – 12, use the graphs of f and g below to evaluate the limits, if they exist.





9. $\lim_{x\to -2}[f(x) + 4\,g(x)]$

$\lim_{x\to-2} f(x) + 4\lim_{x\to-2} g(x)$

$1.5 + 4(-1)$

$\underline{\underline{-2.5}}$

10. $\lim_{x\to 1}[f(x)\cdot g(x)]$

$\lim_{x\to1} f(x) \cdot \lim_{x\to1} g(x)$

$3 \cdot \text{dne}$

$\underline{\underline{\text{DNE}}}$

11. $\lim_{x\to 2}\frac{g(x)}{f(x)}$

$$\frac{\lim_{x\to2} g(x)}{\lim_{x\to2} f(x)} = \frac{0}{4} = \underline{\underline{0}}$$

12. $\lim_{x\to 4}[f(g(x))]^2$

$f(2)=4$

$\left[\lim_{x-4} f(g(x))\right]^2$

$f\left[\lim_{x\to4} g(x)\right]^2 = [f(2)]^2 = \underline{\underline{16}}$

Problems 13 – 16, True or False.

13. If $\lim_{x\to4} f(x) = 2$ and $\lim_{x\to4} g(x) = 0$, then $\lim_{x\to4}\left[\frac{f(x)}{g(x)}\right]$ does not exist.

True

14. If $\lim_{x\to4} f(x) = 0$ and $\mathit{lim}_{x\to4}\, g(x) = 0$, then $\lim_{x\to4}\left[\frac{f(x)}{g(x)}\right]$ does not exist.

False

Factor + cancel, so could be true.

15. $\lim_{x\to1}\frac{x^2+4x-5}{x^2+3x-4} = \frac{\lim_{x\to1} x^2+4x-5}{\lim_{x\to1} x^2+3x-4}$

$$\frac{(x+5)\cancel{(x-1)}}{(x+4)\cancel{(x-1)}}$$

False

16. $\lim_{x\to1}\frac{x^2-3}{x^2+5x-4} = \frac{\lim_{x\to1} x^2-3}{\lim_{x\to1} x^2+5x-4}$

$$\frac{1^2-3}{1^2+5(1)-4} = \frac{1-3}{1^2+5(1)-4}$$

True

Problems 17 – 22, Use the table of values below to find limits for the following:

x	-2	3	6	c
$f(x)$	5	0	1	$\lim_{x\to c} f(x) = 6$
$g(x)$	-3	-2	-4	$\lim_{x\to c} g(x) = -2$
$h(x)$	4	2	3	$\lim_{x\to c} h(x) = 3$

17. $\lim_{x\to c} f(g(x))$

$f\left[\lim_{x\to c} g(x)\right]$

$f(-2)$

$\underline{\underline{5}}$

18. $\lim_{x\to c} g(h(x))$

$g\left[\lim_{x\to c} h(x)\right]$

$g(3)$

$\underline{\underline{-2}}$

19. $\lim_{x\to c} \frac{g(f(x))}{h(g(x))}$

$$\frac{g\left[\lim_{x\to c} f(x)\right]}{h\left[\lim_{x\to c} g(x)\right]} = \frac{g(6)}{h(-2)}$$

$= \underline{\underline{-1}}$

20. $\lim_{x\to c} \frac{f(x)}{f(h(x))}$

$$\frac{\lim_{x\to c} f(x)}{f\left[\lim_{x\to c} h(x)\right]} = \frac{6}{f(3)}$$

$= \frac{6}{0}$ $\underline{\underline{\text{dne}}}$

21. $\lim_{x\to c} f\left(g(h(x))\right)$

$f(g(\lim_{x\to c} h(x))$

$f(g(3))$

$f(-2)$

$\underline{\underline{5}}$

22. $\lim_{x\to c} g\left(h(f(x))\right)$

$g(h(\lim_{x\to c} f(x))$

$g(h(6))$

$g(3)$

$\underline{\underline{-2}}$

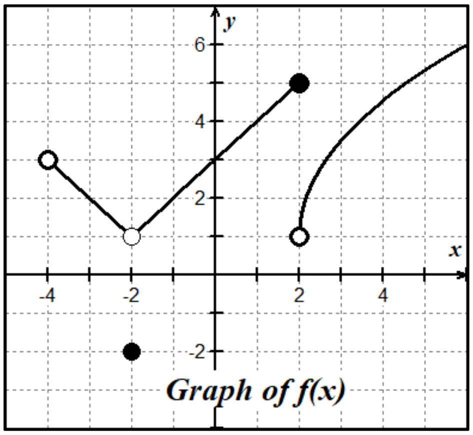
1.4

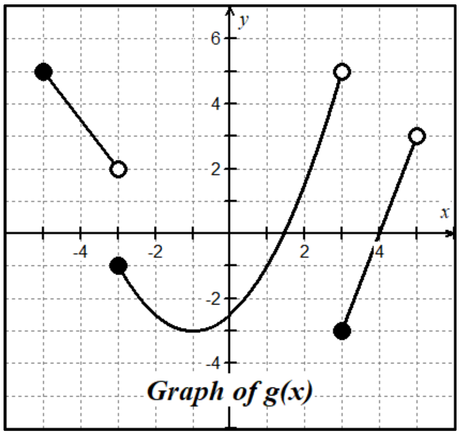
Limits of Composite Functions

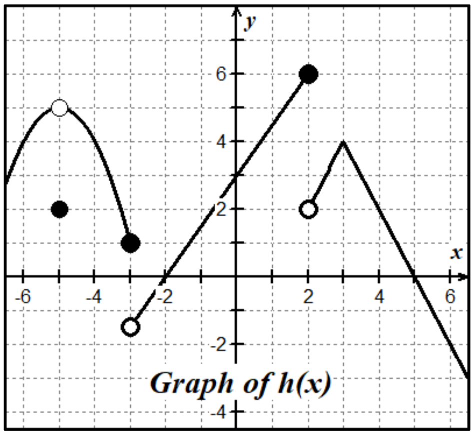
Name ______________________

Date ______________ Period ________

Problems 1 – 8, Use the graphs of $f(x0, g(x)$ and $h(x)$ shown below to answer the following limit questions.







1. $\lim_{x\to -1}[f(x)+g(x)]$

$\lim_{x\to -1} f(x) + \lim_{x\to -1} g(x)$

$2 + (-3) = \underline{\underline{-1}}$

2. $\lim_{x\to 0}[f(x)\cdot h(x)]$

$\lim_{x\to 0} f(x) \cdot \lim_{x\to 0} h(x)$

$(3)(3) = \underline{\underline{9}}$

3. $\lim_{x\to -2} f(h(x))$

Let $u = h(x)$

$\lim_{x\to -2} u = 0$

$\lim_{u\to 0} f(u) = \underline{\underline{3}}$

4. $\lim_{x\to 1} \sqrt{f(x)}$

Let $u = f(x)$

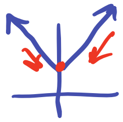
$\lim_{x\to 1} u = 4$

$\lim_{u\to 4} \sqrt{u} = \sqrt{4} = \underline{\underline{2}}$

5. $\lim_{x\to 0} h(|x|+3)$

Let $u = |x|+3$

$\lim_{x\to 0} u = 3^+$



$\lim_{u\to 3^+} u = \underline{\underline{4}}$

6. $\lim_{x\to -5^+}\left[\frac{g(x)}{h(x)}\right]$

$\frac{\lim_{x\to -5^+} g(x)}{\lim_{x\to -5^+} h(x)} = \frac{5}{5} = \underline{\underline{1}}$

7. $\lim_{x\to -5} g(x+2)$

Let $u = x+2$

$\lim_{x\to -5} u = -3$

$\lim_{u\to -3} g(u) \to$ <u>d.n.e.</u>

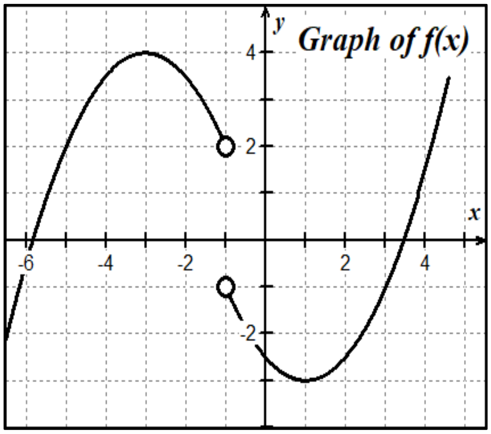
8. $\lim_{x\to 2} f\sqrt{x+3}$

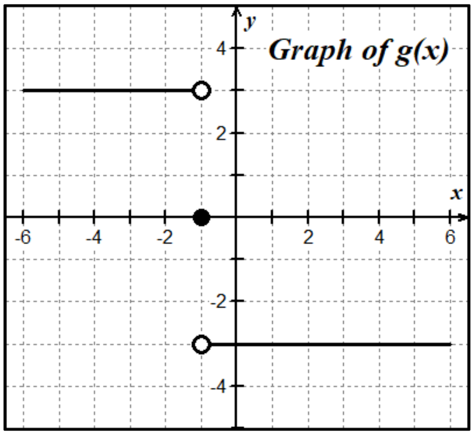
Let $u = \sqrt{x+3}$

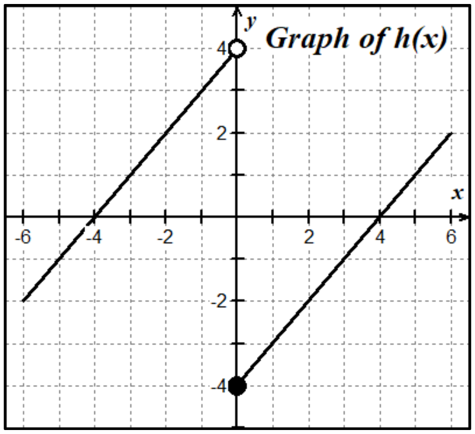
$\lim_{x\to 2} u = \sqrt{1} = 1$

$\lim_{u\to 1} f(u) = \underline{\underline{4}}$

Problems 9 – 14, use the graphs of f, g and h below to evaluate the limits, if they exist.







9. $\lim_{x\to5^+} g(25-x^2)$

$u = 25 - x^2$

$\lim_{x\to5^+} u \to 0^-$

$\lim_{u\to0^-} g(u) = \underline{\underline{-3}}$

10. $\lim_{x\to-1}\left[(g(x))^2-5\right]$

$x\to-1^-$ $\quad$ $x\to-1^+$

$g\to3$ $\quad$ $g\to-3$

$\left.\begin{matrix}(3)^2-5\\(-3)^2-5\end{matrix}\right\}$ same

$\lim_{x\to-1}\left[(g(x))^2-5\right] = \underline{\underline{4}}$

11. $\lim_{x\to0}\frac{h(x)}{h(-x)} = \underline{\underline{-1}}$

let $u=-x$

$x\to0^-$, $h\to4$ $\quad$ $x\to0^+$, $h\to-4$

$\frac{4}{-4}$ or $\frac{-4}{4}$ $\Rightarrow \underline{\underline{-1}}$

12. $\lim_{x\to0} h(h(x)) = \underline{\underline{0}}$

$u = h(x)$

$\lim_{x\to0^-} u = 4^-$

$\lim_{x\to0^+} u = 4^+$

$\left.\begin{matrix}\lim_{u\to4^-} h(u) = 0\\ \lim_{u\to4^+} h(u) = 0\end{matrix}\right\}$ same

13. $\lim_{x\to-1} f(g(x))$

let $u = g(x)$

$\lim_{x\to-1^-} u = 3$

$\lim_{x\to-1^+} u = -3$

$\left.\begin{matrix}\lim_{x\to3} f(u) = -1\\ \lim_{x\to-3} f(u) = 4\end{matrix}\right\}$ not same

$\underline{\underline{\text{limit DNE}}}$

14. $\lim_{x\to-2} f\left(g\left(h(x)\right)\right)$

let $u = h(x)$

$\lim_{x\to-2} u = 2$

let $v = g(u)$

$\lim_{u\to2} v = -3$

$\lim_{v\to-3} f(v) = \underline{\underline{4}}$

Problems 15 – 16, Let $v(x) = \begin{cases} 3, & x = 0 \\ x, & x \neq 0 \end{cases}$ and $w(x) = \begin{cases} -3, & x < 0 \\ 0, & x = 0 \\ 3, & x > 0 \end{cases}$

15. $\lim_{x\to0}\left(v(x)\cdot w(x)\right)$

$\lim_{x\to0^\pm} v(x)\cdot \lim_{x\to0^\pm} w(x) = \underline{\underline{0}}$

$(0)(-3)$ or $(0)(3)$

16. $\lim_{x\to0}\frac{w(x)}{v(x)} = \underline{\underline{\infty}}$

$\lim_{x\to0^-}\frac{w(x)}{v(x)} = \frac{-3}{-0.0001}$ $\quad$ $\lim_{x\to0^+}\frac{w(x)}{v(x)} = \frac{3}{.0001}$

∞ $\quad$ ∞

1.5

Finding Limits by Analytic Methods

Name ______________________________

Date ____________________ Period ________

Problems 1 – 16, Find each of the following limits analytically. Show your algebraic analysis.

1. $\lim_{x\to3}\left(\frac{2}{3}x^2+3x\right)$

$\frac{2}{3}(3)^2+3(3)$

$6+9$

$\underline{\underline{15}}$

2. $\lim_{t\to4}\frac{t-4}{t^2-16}$ direct sub is $\frac{0}{0}$

$\lim_{t\to4}\frac{\cancel{(t-4)}\cdot1}{(t+4)\cancel{(t-4)}}$

$\lim_{t\to4}\frac{1}{t+4}=\underline{\underline{\frac{1}{8}}}$

3. $\lim_{x\to-3}\frac{x^2-5x+6}{2x+6}$ factors: $\frac{(x-3)(x-2)}{2(x+3)}$

check limits from both sides

$x\to-3^+\quad y\to\frac{(+)}{(+)}\Rightarrow\infty$

$x\to-3^-\quad y\to\frac{(+)}{(-)}\Rightarrow-\infty$

D.N.E.

4. $\lim_{\theta\to\pi}\sin^2\theta-3\cos\theta$

$\lim_{\theta\to\pi}(\sin^2\theta)-\lim_{\theta\to\pi}(3\cos\theta)$

$(\sin\pi)^2-3\cos(\pi)$

$0-3(-1)=\underline{\underline{3}}$

5. $\lim_{x\to2}\frac{x^3-8}{x-2}$

$\lim_{x\to2}\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$

$=(2)^2+2(2)+4$

$=\underline{\underline{12}}$

6. $\lim_{\theta\to\frac{\pi}{3}}\frac{\tan\theta}{\theta^2}$

$\frac{\tan\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)^2}=\frac{\sqrt{3}}{\frac{\pi^2}{9}}=\underline{\underline{\frac{9\sqrt{3}}{\pi^2}}}$

7. $\lim_{x\to0}\left[\frac{\sqrt{x+5}-\sqrt{5}}{x}\right]\left[\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}\right]$

$\lim_{x\to0}\frac{\cancel{(x+5-5)}\cdot1}{\cancel{x}(\sqrt{x+5}+\sqrt{5})}$

$=\frac{1}{\sqrt{5}+\sqrt{5}}=\underline{\underline{\frac{1}{2\sqrt{5}}}}$

8. $\lim_{x\to0}\left[\frac{\frac{1}{3+x}-\frac{1}{3}}{x}\right]\left[\frac{3(3+x)}{3(3+x)}\right]$

$\lim_{x\to0}\frac{\cancel{3}-(\cancel{3}+x)}{3x(3+x)}$

$\lim_{x\to0}\frac{-1}{3(3+x)}=\underline{\underline{\frac{-1}{9}}}$

9. $\lim_{x\to e} \frac{3x}{\ln x}$

$$\frac{3e}{\ln e} = \underline{\underline{3e}}$$

10. $\lim_{x\to 2^+} \frac{3x^2+7x+2}{x^2-4}$

$$\lim_{x\to 2+} \frac{(3x+1)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$$

$$\frac{3(2.0001)+1}{2.0001-2} = \frac{7.0003}{0.0001} \Rightarrow \underline{\underline{\infty}}$$

11. $\lim_{x\to 3} \left[\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\right]\left[\frac{3x}{3x}\right]$

$$\lim_{x\to 3} \frac{3-x}{3x(x-3)}$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}} = \underline{\frac{-1}{9}}$$

12. $\lim_{x\to \frac{3}{2}} \frac{8x^3-27}{2x-3}$

$$\lim_{x\to 3/2} \frac{\cancel{(2x-3)}(4x^2+6x+9)}{\cancel{(2x-3)}}$$

$$4\left(\tfrac{3}{2}\right)^2+6\left(\tfrac{3}{2}\right)+9$$

$$9 + 9 + 9$$

$$\underline{27}$$

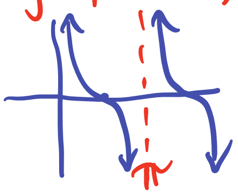
13. $\lim_{x\to \pi^+} \cot x$

as $x\to\pi^+$

$y\to\infty$

$\underline{\underline{\infty}}$

graphically



14. $\lim_{x\to 0} \cos(x+\sin x)$

$$\cos(0+\sin 0)$$

$$\cos(0+0)$$

$$\cos(0)$$

$$\underline{\underline{1}}$$

15. $f(x) = \begin{cases} 3x-1, & x\le 1 \\ 3x^2, & x>1 \end{cases}$, find $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1^-} f(x) = 3(1)-1 \Rightarrow 2$$

$$\lim_{x\to 1^+} f(x) = 3(1)^2 \Rightarrow 3$$

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

∴ limit D.N.E.

16. $\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

$$\lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$\lim_{h\to 0} (2x+h) = \underline{\underline{2x}}$$

Problems 17 – 32, find the limits, if they exist, for the given piecewise-defined functions.

$$f(x)=\begin{cases} x+3, & x<2 \\ x^2-1, & 2\le x<4 \\ \sqrt{x+5}, & x\ge 4 \end{cases}$$

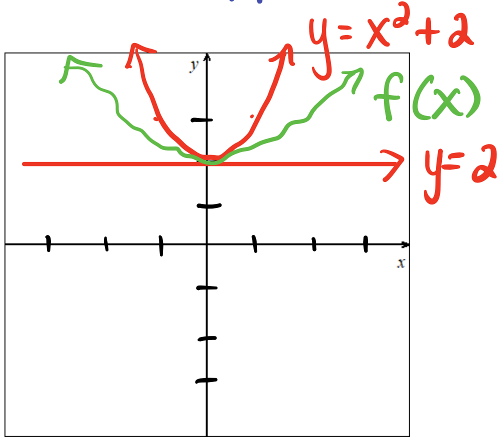
$$g(x)=\begin{cases} 4-x^2, & x\le 1 \\ 7, & 1<x<3 \\ 2-x, & x\ge 3 \end{cases}$$

17. $\lim_{x\to 2^+} f(x)$ 3	**18.** $\lim_{x\to 2^-} f(x)$ 5	**19.** $\lim_{x\to 2} f(x)$ DNE	**20.** $f(2)$ 3
21. $\lim_{x\to 4^+} f(x)$ 3	**22.** $\lim_{x\to 4^-} f(x)$ 15	**23.** $\lim_{x\to 4} f(x)$ DNE	**24.** $f(4)$ 3
25. $\lim_{x\to 1^-} g(x)$ 3	**26.** $\lim_{x\to 1^+} g(x)$ 7	**27.** $\lim_{x\to 1} g(x)$ DNE	**28.** $g(1)$ 3
29. $\lim_{x\to 3^-} g(x)$ 7	**30.** $\lim_{x\to 3^+} g(x)$ -1	**31.** $\lim_{x\to 3} g(x)$ DNE	**32.** $g(3)$ -1

33. If $2\le f(x)\le x^2+2$ for all x, find $\lim_{x\to 0} f(x)$. Sketch a graph to illustrate.

$y=2$ is lower bound

$y=x^2+2$ is upper bound



34. If $\lim_{x\to c} f(x)=-5$ and $\lim_{x\to c} g(x)=8$, find $\lim_{x\to c}\dfrac{2f(x)}{g(x)-f(x)}$

$$\frac{2\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)-\lim_{x\to c} f(x)}$$

$$\frac{2(-5)}{8-(-5)}=\frac{-10}{13}$$

1.6 Limits of Transcendental Functions

Name ____________________

Date ____________________ Period ________

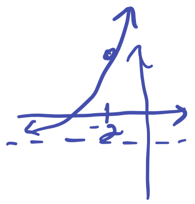
Problems 1 – 6, find the limits of each exponential function. If necessary, sketch a graph of the function to help you determine the behavior and limit.

1. $\lim_{x\to -2} 3^{x+4} - 2 = \underline{\underline{7}}$

$3^2 - 2$

b>1 growth

above y=-2

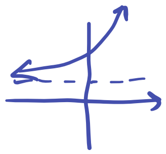


2. $\lim_{x\to\infty} 2^{x-3} + 4 = \underline{\underline{\infty}}$

b>1 growth

increasing

as x→∞



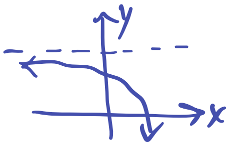
3. $\lim_{x\to -\infty} -(0.25)^{-x+1} + 5 = \underline{\underline{5}}$

$\left(\frac{1}{4}\right)^{-1} = 4$

b>1 growth

a<0 reflects

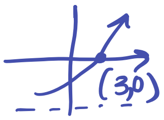
decreasing



4. $\lim_{x\to 3} \left(\frac{1}{2}\right)^{3-x} - 1 = \underline{\underline{0}}$

$\left(\frac{1}{2}\right)^{-1} \Rightarrow 2$

b>1 growth



5. $\lim_{x\to -1} -4^{-x-1} + 5 = \underline{\underline{4}}$

$(4)^{-1} = \frac{1}{4}$

b<1 decay

a<0 reflects

increasing

$-4^{1-1}+5$

-4^0+5

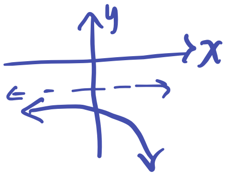
6. $\lim_{x\to\infty} -\left(\frac{1}{3}\right)^{2-x} - 3 = \underline{\underline{-\infty}}$

$\left(\frac{1}{3}\right)^{-1} \Rightarrow 3$

b>1 growth

a<0 reflects

decreasing



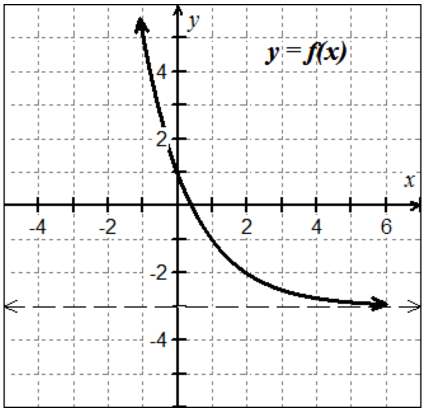
Problems 7 – 10, use the graph of $y = f(x)$ at right, to find each of the following limits.

7. $\lim_{x\to\infty} f(x) = \underline{-3}$

8. $\lim_{x\to -\infty} f(x) = \underline{\infty}$

9. $\lim_{x\to 0} f(x) = \underline{1}$

10. $\lim_{x\to 2} f(x) = \underline{-2}$



Problems 11 – 28, evaluate the limits, if they exist.

11. $\lim_{x\to 0} \frac{\sin\ 3x}{x}$

$$\lim_{x\to 0}\left[\frac{\sin 3x}{x}\right]\left[\frac{3}{3}\right]$$

$$\lim_{x\to 0} 3 \cdot \lim_{x\to 0}\left[\frac{\sin 3x}{3x}\right]$$

$$\underline{\underline{3}}$$

12. $\lim_{x\to 0} \frac{\sin x \sec x}{x}$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot \lim_{x\to 0}\left(\frac{1}{\cos x}\right)$$

$$1 \cdot 1$$

$$\underline{\underline{1}}$$

13. $\lim_{x\to 0} \frac{e^x\ \cos x}{2}$

$$\frac{e^0 \cos(0)}{2}$$

$$\underline{\underline{\frac{1}{2}}}$$

14. $\lim_{x\to \frac{\pi}{2}} \frac{1\ -\ \cos x}{3x}$

$$\frac{1-\cos\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{1-0}{\frac{3\pi}{2}}$$

$$= \underline{\underline{\frac{2}{3\pi}}}$$

15. $\lim_{x\to 0} \frac{x\ +\ \sin x}{x}$

$$\lim_{x\to 0}\left(\frac{x}{x}\right) + \lim_{x\to 0}\left(\frac{\sin x}{x}\right)$$

$$\lim_{x\to 0}(1) + 1$$

$$\underline{\underline{2}}$$

16. $\lim_{x\to 4}\ e^{3-x} + 4$

$$e^{3-4} + 4$$

$$e^{-1} + 4$$

$$\underline{\underline{\frac{1+4e}{e}}}$$

17. $\lim_{\theta\to \frac{\pi}{3}} \frac{\cos^2\theta}{1\ -\ \sin\theta}$

$$\frac{\cos\left(\frac{\pi}{3}\right)^2}{1-\sin\left(\frac{\pi}{3}\right)} = \frac{\frac{1}{4}}{1-\frac{\sqrt{3}}{2}}$$

$$\frac{1}{4}\left[\frac{2}{2-\sqrt{3}}\right] \Rightarrow \underline{\underline{\frac{1}{2(2-\sqrt{3})}}}$$

18. $\lim_{x\to 0} \frac{\sin x}{3x^2 - x}$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot \lim_{x\to 0}\left[\frac{1}{3x-1}\right]$$

$$1 \cdot (-1)$$

$$\underline{\underline{-1}}$$

19. $\lim_{x\to 2} \begin{cases} 3x^2-2x, & x<2 \\ 7-\cos\left(\frac{\pi x}{3}\right), & x\ge 2\end{cases}$

$x\to 2^-$ $\quad 3(2)^2-2(2)=8$

$x\to 2^+$ $\quad 7-\cos\left(\frac{2\pi}{3}\right)=7.5$

$\therefore \underline{\underline{\lim_{x\to 2} f(x) \text{ dne}}}$

20. $\lim_{x\to 3} \begin{cases} \ln x, & 0<x<2 \\ x^2\ln x, 2\le x\le 4\end{cases}$

$\lim_{x\to 3}\left[x^2\cdot \ln x\right]$

$3^2\cdot \ln 3$

$\underline{\underline{9\ln(3)}}$

21. $\lim_{x\to\frac{\pi}{3}} \frac{2\sin^2 x + 3\sin x - 2}{2\sin x - 1}$

$\lim_{x\to\pi/3} \frac{\cancel{(2\sin x-1)}(\sin x+2)}{\cancel{(2\sin x-1)}}$

$\sin(\pi/3)+2$

$\underline{\underline{\frac{\sqrt{3}}{2}+2}}$

22. $\lim_{x\to 0}\frac{5x+\sin 2x}{x}$

$\lim_{x\to 0}\left(\frac{5x}{x}\right)+\lim_{x\to 0}\left[\frac{\sin 2x}{x}\right]\left[\frac{2}{2}\right]$

$\lim_{x\to 0}(5)+\lim_{x\to 0}\left[2\left(\frac{\sin \cancel{2x}}{\cancel{2x}}\right)\right]$

$5+2=\underline{\underline{7}}$

23. $\lim_{x\to 0}\frac{3^x\cos x}{4}$

$\frac{3^0(\cos 0)}{4}=\underline{\underline{\frac{1}{4}}}$

24. $\lim_{x\to -3}\left(\frac{1}{4}\right)^{-x-4}+5$

$\left(\frac{1}{4}\right)^{3-4}+5$

$4+5$

$\underline{\underline{9}}$

25. $\lim_{\theta\to 0}\frac{5\cos\theta-5}{\theta}$

$\lim_{\theta\to 0}\frac{5(\cos\theta-1)}{\theta}$

$\lim_{\theta\to 0}5\cdot\lim_{\theta\to 0}\left[\frac{\cos\theta-1}{\theta}\right]$

$5(0)=\underline{\underline{0}}$

26. $\lim_{x\to -3}\frac{(x+3)\ln(x+5)}{x^2-9}$

$\lim_{x\to -3}\frac{\cancel{(x+3)}\ln(x+5)}{\cancel{(x+3)}(x-3)}$

$\frac{\ln(-3+5)}{-3-3}=\underline{\underline{-\frac{1}{6}\ln 2}}$

27. $\lim\limits_{\theta\to0}\dfrac{\sin 3\theta\ +\ 1\ -\ \cos\theta}{\theta}$

$$\lim_{\theta\to0}\left(\frac{\sin 3\theta}{\theta}\right)\left(\frac{3}{3}\right)+\cancelto{0}{\lim_{\theta\to0}\left[\frac{1-\cos\theta}{\theta}\right]}$$

$$\lim_{\theta\to0} 3\left[\frac{\sin 3\theta}{3\theta}\right]=\underline{\underline{3}}$$

28. $\lim\limits_{\theta\to0}\dfrac{\cos\theta\sin\theta-\sin\theta}{\theta^2}$

$$\lim_{\theta\to0}\frac{\sin\theta(\cos\theta-1)}{\theta\cdot\theta}$$

$$\lim_{\theta\to0}\left[\frac{\sin\theta}{\theta}\right]\cdot\lim_{\theta\to0}\left[\frac{\cos\theta-1}{\theta}\right]$$

$$1\cdot0$$

$$\underline{\underline{0}}$$

Problems 29 – 30, Use the Squeeze Theorem to evaluate the following limits.

29. $\lim\limits_{x\to0}x^2\cos\left(\dfrac{1}{x}\right)$

know $-1\le\cos x\le1$ and $x^2\ge0$

$$-1\le\cos\left(\tfrac{1}{x}\right)\le1$$

$$-x^2\le x^2\cos\left(\tfrac{1}{x}\right)\le x^2$$

$$\lim_{x\to0}(-x^2)=0 \quad\text{and}\quad \lim_{x\to0}x^2=0$$

$\therefore$ by Squeeze Theorem, $\lim\limits_{x\to0}x^2\cos\left(\tfrac{1}{x}\right)=0$

30. If $0\le f(x)\le c$ for some real number c, prove that $\lim\limits_{x\to0}x^2f(x)=0$

Given $0\le f(x)\le c$, we know $x^2\ge0$, so

$$0\le x^2\cdot f(x)\le cx^2$$

$$\lim_{x\to0}0=0 \quad\text{and}\quad \lim_{x\to0}cx^2=0$$

$\therefore$ by Squeeze Theorem, $\lim\limits_{x\to0}x^2f(x)=0$

1.7

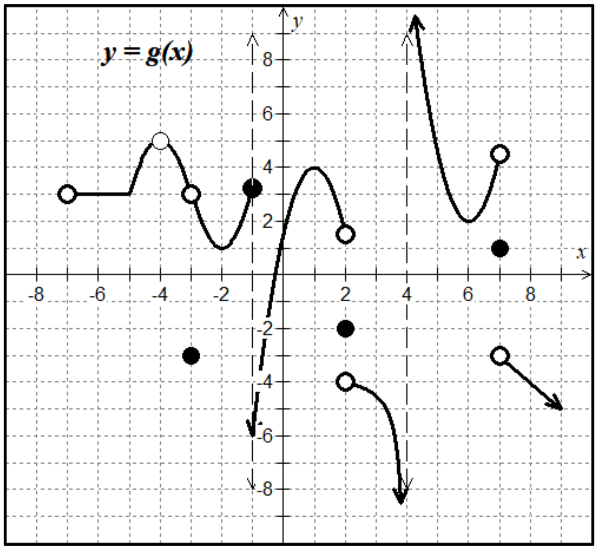
Limits and Continuity

Name ______________________________

Date ____________________ Period ________

1. Use the graph of the function $y = g(x)$ shown below, to evaluate each of the following.

A. $\lim_{x\to-3} g(x) =$ 3	**B.** $\lim_{x\to-1^-} g(x) =$ 3
C. $\lim_{x\to-1^+} g(x) =$ $-\infty$	**D.** $g(-1) =$ 3
E. $\lim_{x\to-1} g(x) =$ dne	**F.** $\lim_{x\to4^+} g(x) =$ $+\infty$
G. $\lim_{x\to4^-} g(x) =$ $-\infty$	**H.** $\lim_{x\to7} g(x) =$ dne
J. $g(7) =$ 1	**K.** $\lim_{x\to2^+} g(x) =$ -4



2. Use the graph of the function $y = f(x)$ shown below, to evaluate each of the following.

A. On the interval $x \in (-\infty, \infty)$, list the largest intervals for which $f(x)$ is continuous.

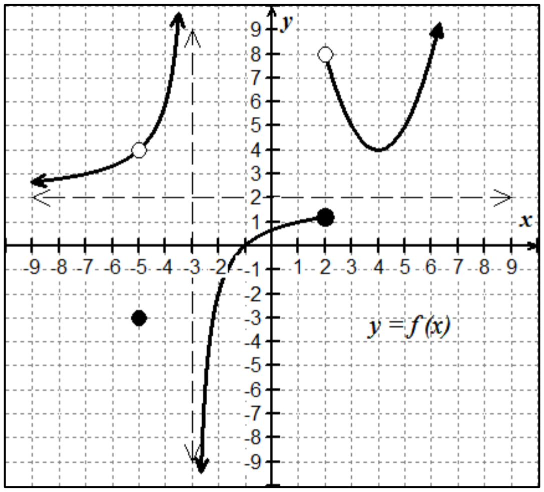
$(-\infty, -5)\ (-5, -3)\ (-3, 2]\ (2, \infty)$

B. Find the smallest value k, such that the function is continuous on (k, ∞)

$k = 2$, then f is continuous on $2 < x < \infty$

C. Find the smallest value k, such that the function is continuous on $[k, \infty)$

Since $k > 2$ on $f(x)$ there is no such smallest k-value.



D. Find the largest value of b such that $y = f(x)$ is continuous on $(-3, b]$ but not continuous in $(-3, b + 1]$. State all values of b that would work.

$b = 2$; $(-3, 2]$ continuous but $(-3, 2+1]$ discontinuous

all b-values: $1 < b \leq 2$

Problem 3 – 6, determine the points, classify the type for each as removable, non-removable, jump, or infinite.

3. $f(x) = \frac{1}{(x-3)^2}$

$x=3$ infinite discontinuity
non-removable
(vertical asymptote)

4. $g(x) = \frac{x-4}{x^2-9x+20}$

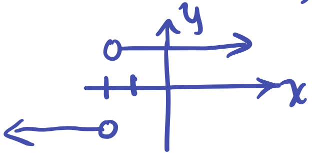
$\frac{1\cancel{(x-4)}}{(x-5)\cancel{(x-4)}}$

$x=4$ removable (hole)

$x=5$ infinite discontinuity
non-removable
(vertical asymptote)

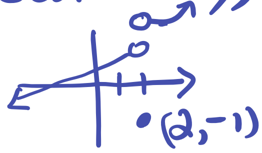
5. $h(x) = \frac{|x+2|}{x+2}$

$x=-2$ non-removable
(jump discontinuity)



6. $f(x) = \begin{cases} x+1 & x<2 \\ -1 & x=2 \\ x^2+1 & x>2 \end{cases}$

$x=2$ non-removable
(jump discontinuity)



Problems 7 – 8, use the three-part definition of continuity to determine if the given functions are continuous at the indicated values of *x*.

7. $f(x) = \begin{cases} e^x \cos x\,, & x \geq \pi \\ e^x \tan\left(\frac{3x}{4}\right), & x < \pi \end{cases}$ at $x = \pi$

1) $f(\pi) = e^{\pi}\cos\pi = -e^{\pi}$

2) $\lim\limits_{x\to\pi^-} e^x \tan\left(\frac{3x}{4}\right) = -e^{\pi}$

$\lim\limits_{x\to\pi^+} e^x \cos x = -e^{\pi}$

3) $f(x) = \lim\limits_{x\to\pi} f(x)$

$\therefore f(x)$ is continuous at $x=\pi$.

8. $g(x) = \begin{cases} \frac{x^2-9}{x+3} & x \neq -3 \\ 5 & x = -3 \end{cases}$ at $x = -3$

1) $g(-3) = 5$

2) $\lim\limits_{x\to -3^-} g(x) = -6$

$\lim\limits_{x\to -3^+} g(x) = -6$

3) $\lim\limits_{x\to -3} g(x) \neq g(-3)$

$\therefore g(x)$ is not continuous at $x=-3$.

Problems 9 – 12, find all value(s) of *a, b, c* or *k* that make the function continuous everywhere.

9. $f(x) = \begin{cases} kx^2 & x \le 3 \\ 4x - 11 & x > 3 \end{cases}$

$9k = 4(3) - 11$

$9k = 1$

$\underline{\underline{k = 1/9}}$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x)$

when $k = 1/9$, then

$\lim\limits_{x \to 3} f(x) = 1$ and $f(3) = 1$

10. $g(x) = \begin{cases} cx^2 & x < 1 \\ 4 & x = 1 \\ -x^3 + kx & x > 1 \end{cases}$

$c = 4 \quad -1 + k = 4$

$\underline{\underline{k = 5}}$

$\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^+} g(x)$

when $c = 4$ and $k = 5$.

Then, $\lim\limits_{x \to 1} g(x) = 4$ and $g(1) = 4$.

11. $h(x) = \begin{cases} \pi & x < 0 \\ x^2 + ax + b & 0 \le x \le 1 \\ 6x + 5 & x > 1 \end{cases}$

$\pi = 0^2 + 0 \cdot a + b \qquad x = 0$

$\underline{b = \pi}$

$1 + a + \pi = 6(1) + 5$

$a + \pi = 10$

$\underline{\underline{a = 10 - \pi}} \qquad x = 1$

$\therefore \lim\limits_{x \to 0} h(x) = \pi$ and

$\lim\limits_{x \to 1} h(x) = 11$ when

$\underline{\underline{a = 10 - \pi \text{ and } b = \pi}}$

12. $f(x) = \begin{cases} x^2 & x < 1 \\ \sin(bx) & x \ge 1 \end{cases}$

$\sin b = 1 \qquad x = 1$

$\sin^{-1} 1 = b$

$b = \pi/2$

$\therefore \lim\limits_{x \to 1} f(x) = 1$ when

$b = \frac{\pi}{2}$. Then,

$\lim\limits_{x \to 1} f(x) = 1$ and $f(1) = 1$

13. Consider the function $y = f(x)$ to answer the following. $f(x) = \begin{cases} -3 & x \le -1 \\ mx + k & -1 < x < 4 \\ 3 & x \ge 4 \end{cases}$

A. What two limits must be equal in order for the function to be continuous at $x = -1$?

$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x)$ then $mx + k = -3$

$-m + k = -3$

B. What two limits must be equal in order for the function to be continuous at $x = 4$?

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^+} f(x)$ then $mx + k = 3$

$4m + k = 3$

C. Find the values of m and k so that the function is continuous everywhere.

$$\begin{cases} 4m + k = 3 \\ -m + k = -3 \end{cases} \Rightarrow \begin{array}{r} 4m + k = 3 \\ m - k = 3 \\ \hline 5m = 6 \end{array}$$

$m = 6/5$

$-\left(\frac{6}{5}\right) + k = -3$

$k = \frac{-15 + 6}{5}$

$k = -9/5$

$m = \frac{6}{5}, \; k = -\frac{9}{5}$

14. If $y = f(x)$ is continuous for all $x \ne \frac{1}{2}$, evaluate the following. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{2x^2 + 3x - 2}, & x \ne -2 \\ k, & x = -2 \end{cases}$

$f(x) = \dfrac{(x-3)\cancel{(x+2)}}{(2x-1)\cancel{(x+2)}}$

A. $\lim\limits_{x \to \frac{1}{2}^+} f(x) = -\infty$

$\lim\limits_{x \to 1/2^+} \left(\dfrac{x-3}{2x-1}\right)$

$\dfrac{.5001 - 3}{2(.5001) - 1} = \dfrac{-2.4999}{0.0002}$

$= -\infty$

B. $\lim\limits_{x \to 1} f(x) = -2$

$\lim\limits_{x \to 1} \left(\dfrac{x-3}{2x-1}\right)$

$\dfrac{1 - 3}{2(1) - 1} = -2$

C. What is the value of k ?

@ $x = -2$

$\dfrac{-2 - 3}{2(-2) - 1} = k$

$\dfrac{-5}{-5} = k$

$k = 1$

1.8

Infinite Limits and Limits at Infinity

Name ______________________________

Date ____________________ Period ________

Problem 1 – 4, Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, then identify all horizontal or slant asymptotes.

1. $f(x) = \dfrac{7+2x-5x^2}{2x^2-7x-4}$

$\lim_{x\to\infty} f(x) = -\frac{5}{2}$

$\lim_{x\to-\infty} f(x) = -\frac{5}{2}$

H.A. @ $y = -\frac{5}{2}$

2. $f(x) = \dfrac{2x-3}{\sqrt{4x^2+3}}$

$\frac{2x}{|x|}$

$\lim_{x\to\infty} f(x) = 1$

$\lim_{x\to-\infty} f(x) = -1$

H.A. @ $y = 1$ and $y = -1$

3. $f(x) = \dfrac{3+4x-2x^3}{x^2+1}$

$\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to\infty} f(x) = -\infty$

$$\begin{array}{r} -2x \\ x^2+1\overline{)-2x^3+4x+3} \\ 2x^3+2x \end{array}$$

S.A. @ $y = -2x$

4. $f(x) = \dfrac{x^{4/3}+x^{1/3}}{(4x^{2/3}+1)^2}$

$\frac{x^{4/3}}{16x^{4/3}}$

$\lim_{x\to\infty} f(x) = \frac{1}{16}$

$\lim_{x\to-\infty} f(x) = \frac{1}{16}$

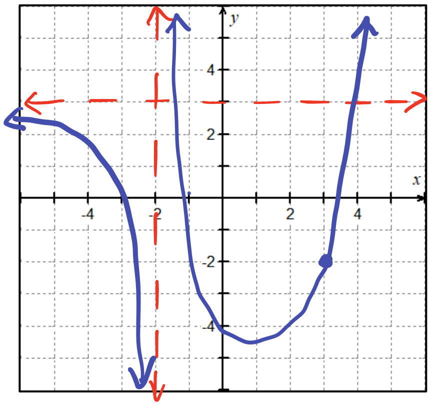
HA. @ $y = \frac{1}{16}$

Problems 5 – 6, sketch a function that satisfies the stated conditions. Include asymptotes.

5. $\lim_{x\to 3} f(x) = -2$ $\quad \lim_{x\to -2^-} f(x) = -\infty$

$\lim_{x\to -2^+} f(x) = +\infty$ $\quad \lim_{x\to -\infty} f(x) = 3$

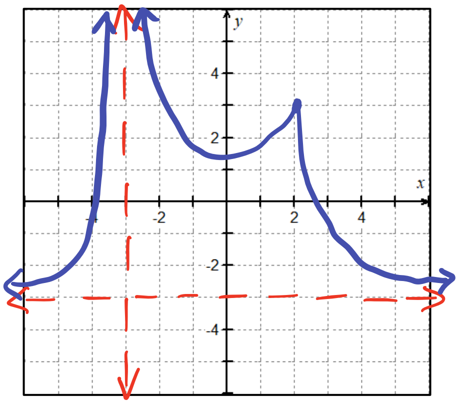
$\lim_{x\to +\infty} f(x) = \infty$



6. $\lim_{x\to -3^-} f(x) = +\infty$ $\quad \lim_{x\to -3^+} f(x) = +\infty$

$\lim_{x\to 2} f(x) = 3$ $\quad \lim_{x\to -\infty} f(x) = -3$

$\lim_{x\to +\infty} f(x) = -3$



Problems 7 – 10, Identify all vertical asymptotes and find $\lim_{x\to a^+} f(x)$; $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a} f(x)$, where a is the x-value of the asymptote.

7. $f(x) = \frac{x+4}{x^2+9x+20} \Rightarrow \frac{1}{x+5}$

V.A. $x=-5$

$\lim_{x\to -5^-} f(x) = -\infty$

$\lim_{x\to -5^+} f(x) = \infty$

$\lim_{x\to -5} f(x)$ dne

x	f(x)
-5.001	(−)
-4.999	(+)

8. $f(x) = \frac{2x^2-x-15}{x^2-5x+6} \Rightarrow \frac{2x+5}{x-2}$

V.A $x=2$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = \infty$

$\lim_{x\to 2} f(x)$ dne

x	f(x)
1.999	+/−
2.001	+/+

9. $f(x) = \frac{\ln(x^2+1)}{x+1}$

V.A. $x={}^-1$

$\lim_{x\to {}^-1^-} f(x) = -\infty$

$\lim_{x\to {}^-1^+} f(x) = \infty$

$\lim_{x\to {}^-1} f(x)$ dne

x	f(x)
-1.001	+/−
-0.999	+/+

10. $f(x) = \frac{x^2+3x-18}{x^2-6x+9} \Rightarrow \frac{x+6}{x-3}$

VA $x=3$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to 3^+} f(x) = \infty$

$\lim_{x\to 3} f(x)$ dne

x	f(x)
2.999	+/−
3.001	+/+

Problems 11 – 12, For the piecewise functions, find the limit as $x \to \infty$, $x \to -\infty$, $x \to 0^-$, and $x \to 0^+$.

11. $f(x) = \begin{cases} \frac{2x+3}{x-1}, & x < 0 \\ \frac{1}{x}, & x \geq 0 \end{cases}$

$\lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to-\infty} \frac{2x+3}{x-1} = 2$

$\lim_{x\to 0^-} \frac{2x+3}{x-1} = -3$

$\lim_{x\to 0^+} \frac{1}{x} = \infty$

12. $g(x) = \begin{cases} \frac{1}{x^2}, & x < 0 \\ \frac{2x}{x+1}, & x \geq 0 \end{cases}$

$\lim_{x\to\infty} \frac{2x}{x+1} = 2$

$\lim_{x\to-\infty} \frac{1}{x^2} = 0$

$\lim_{x\to 0^-} \frac{1}{x^2} = \infty$

$\lim_{x\to 0^+} \frac{2x}{x+1} = 0$

Problems 13 – 20, Evaluate the following limits without the aid of a calculator.

13. $\lim_{x\to\infty}\left[\left(\frac{3x^2-1}{x^2}\right)\left(\frac{2}{x}-1\right)\right]$

$$\lim_{x\to\infty}\left(\frac{3x^2-1}{x^2}\right)\left[\lim_{x\to\infty}\left(\frac{2}{x}\right)-\lim_{x\to\infty}1\right]$$

$$3\,(0-1)$$

$$\underline{\underline{-3}}$$

14. $\lim_{x\to-\infty}\left(\frac{5+4x-3x^2}{2x^2+1}\right)$

by end-behavior model

$$\frac{-3x^2}{2x^2}$$

$$\underline{\underline{\lim_{x\to-\infty}f(x)=\frac{-3}{2}}}$$

15. $\lim_{x\to\infty}(e^{-x}\sin x)$

$$\lim_{x\to\infty}\left(\frac{\sin x}{e^x}\right)=0$$

$$\sin x \quad -1\le x\le 1$$

$$x\to\infty \quad e^x\to\infty$$

16. $\lim_{x\to3^+}\ln(x^2-9)$

$$\lim_{x\to3^+}\ln[(x+3)(x-3)]$$

$$\underline{\underline{\lim_{x\to3^+}f(x)=-\infty}}$$

x	$f(x)$
3.001	−

17. $\lim_{x\to-1^-}\frac{x+1}{x^4-1}$

$$\frac{\cancel{(x+1)}}{(x^2+1)\cancel{(x+1)}(x-1)}$$

$$\lim_{x\to-1^-}\frac{1}{(x^2+1)(x-1)}=\underline{\underline{-\frac{1}{4}}}$$

18. $\lim_{x\to0^-}\frac{2^x}{x}$

$$\lim_{x\to0^-}\frac{2^x}{x}=\underline{\underline{-\infty}}$$

$$\frac{1}{2^{0.1}(-0.1)}\to\frac{(+)}{(-)}$$

19. $\lim_{x\to\infty}\frac{x^2+3x}{\sqrt{2x^2+1}}$

divide by $\sqrt{x^2}$

$$\lim_{x\to\infty}\frac{x+3}{\sqrt{2+\frac{1}{x^2}}}\Rightarrow\frac{\infty+3}{\sqrt{2+0}}$$

$$\lim_{x\to\infty}f(x)=\underline{\underline{\infty}}$$

20. $\lim_{x\to\infty}\frac{4x+9}{2x^2-x+6}$

by end-behavior model

$$\frac{4x}{2x^2}$$

$$\lim_{x\to\infty}f(x)=\underline{\underline{0}}$$

Problems 21 – 26, Use the graph of $y = g(x)$ at right to find the limits.

21. $\lim_{x \to -3^-} g(x) =$ ∞

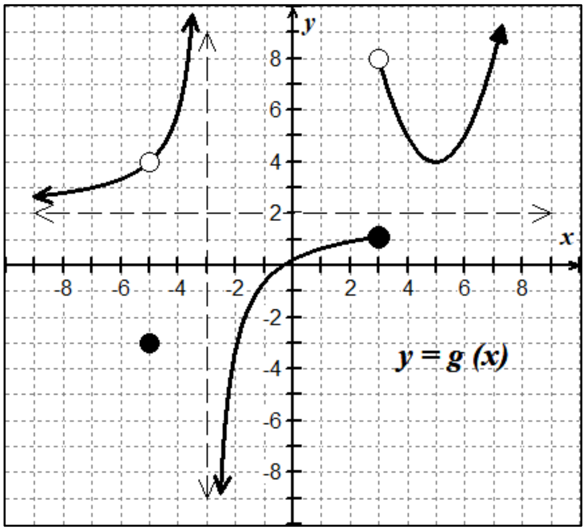
22. $\lim_{x \to -3^+} g(x) =$ $-\infty$

23. $\lim_{x \to -3} g(x) =$ dne

24. $\lim_{x \to -\infty} g(x) =$ 2

25. $\lim_{x \to \infty} g(x) =$ ∞

26. $\lim_{x \to 3^+} g(x) =$ 8



Problems 27 – 29, Multiple Choice.

C **27.** Find the limit: $\lim_{x \to -\infty} \frac{4+e^{-x}}{1-e^{-x}}$

(A) 4 **(B)** -4 [✓ **(C)** -1] **(D)** $-\infty$

EBM $\frac{e^{-x}}{-e^{-x}} = -1$

B **28.** Find the limit: $\lim_{x \to 3} \left(2 - \frac{5}{(x-3)^2}\right)$

(A) ∞ [✓ **(B)** $-\infty$] **(C)** 2 **(D)** -3

$\lim_{x \to 3} 2 - \lim_{x \to 3} \frac{5}{(x-3)^2}$ $2 - \infty = -\infty$

B **29.** $f(x)$ decreases without bound as x approaches what value from the right?

$f(x) = \frac{6}{(x-3)(7-x)}$

(A) -7 [✓ **(B)** 7] **(C)** -3 **(D)** 3

$\lim_{x \to 7^+} f(x) = \frac{6}{(7.01-3)(7-7.01)}$ $\frac{(+)}{(-)}$ decreases

1.q

Intermediate Value Theorem

Name ______________________________

Date ____________________ Period _________

Problems 1 – 13, Use the Intermediate Value Theorem to complete.

1. In the function $f(x) = x^3 - x - 1$, it can be shown that $f(1) = -1$ and $f(2) = 5$. Complete the table below to find an approximation for a solution of the interval $[1, 2]$.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$f(x)$	−1	−0.77	−0.47	−0.10	0.34	0.88	1.5	2.2	3.0	3.9	5

A solution occurs on $1.3 \leq x \leq 1.4$ such that $f(x)=0$.

2. Find the value of c guaranteed by the Intermediate Value Theorem. $f(x) = x^2 + 4x - 13$ on $[0, 4]$ such that $f(c) = 8$

1.) $f(x)$ is continuous over reals

2) $f(0)=-13$; $f(4)=19$

3) $(x+7)(x-3)=0$

$x \neq -7 \quad x=3$

4) $f(0)<8$ and $f(4)>8$

by IVT, there exists $x=c$ where $f(c)=8$

At $c=3$ $f(3)=8$ on $0 \leq x \leq 4$.

3. Show that $g(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval $[-1, 2]$.

1.) Polynomial functions are continuous everywhere.

2) $g(-1)=8$ and $g(2)=-19$

3) By IVT, there exists a value c, such that $g(c)=0$ since $-19<0<8$ on $-1 \leq c \leq 2$.

4. Between which of the following two values does the equation $3x^3 + 5x - 11 = 0$ have a solution?

(A) $[-2, -1]$	**(B)** $[0, 1]$	**(C)** $[-1, 0]$	$\boxed{\textbf{(D)}\ [1, 2]}$
$f(-2)=-45$	$f(0)=-11$	$f(-1)=-19$	$f(1)=-3$
$f(-1)=-19$	$f(1)=-3$	$f(0)=-11$	$f(2)=23$

5. Given the function $f(x) = \frac{2x-3}{2x-5}$, determine which interval(s) satisfies the conditions for the Intermediate Value Theorem, such that $f(x) = 0$.

(A) One solution between x = 0 and x = 1

[✓ **(B)** One solution between $x = 1$ and $x = 2$]

(C) One solution between $x = 1$ and $x = 2$ and one solution between $x = 2$ and $x = 3$

(D) One solution between $x = 2$ and $x = 3$

$f(x)$ is not continuous at $x = 5/2$

$f(0) = 3/5$ $\underline{\underline{f(1) = 1/3}}$ $\underline{\underline{f(2) = -1}}$ $f(3) = 3$

---

6. Apply the Intermediate Value Theorem, if possible, on [1, 2] so that $f(c) = 9$ for the function $f(x) = x^3 + x$.

1) $f(x)$ is continuous on reals
2) $f(1) = 2$; $f(2) = 10$
3) $2 < 9 < 10$, so by IVT, there exists a value $x = c$ where $f(c) = 9$ on $1 \leq x \leq 2$.

---

7. A delivery van travels along a straight road. During the time interval $0 \leq t \leq 30$ seconds, the van's velocity in feet per second is a continuous function. Use the table below to find the minimum number of times that the van must have been stopped. Justify your answer.

***t* (sec)**	0	5	7	12	18	22	30
$V(t)$ (ft/sec)	−28	−60	−15	8	24	−4	10

By IVT, the van is stopped at least 3 times

1) on [7, 12] $-15 < V(c) < 8$
2) on [18, 22] $-4 < V(c) < 24$
3) on [22, 30] $-4 < V(c) < 10$

$*$ V(t) is a continuous function

---

8. Explain why the Intermediate Value Theorem does not apply for guaranteeing that a zero exists for the function $f(x) = x^2 + 2x + 5$ over [0, 6].

1) $f(x)$ is continuous on reals
2) $f(0) = 5$ and $f(6) = 53$
3) Since $f(x) > 0$ for entire interval, no zero is possible on $0 \leq x \leq 6$, IVT does not apply

9. The functions f and g are continuous. The function h is given by $h(x) = f(g(x)) - x$. The table below gives values of the functions. Explain why there must be a value c for $1 < c < 5$ such that $h(c) = -2$.

x	1	2	3	4	5
$f(x)$	0	9	7	-3	8
$g(x)$	4	6	-4	1	3

1) f and g are continuous, so h(x) will be continuous on [1, 5] too.

2) $h(1) = f(g(1))-1$
$= f(4)-1$
$h(1) = -4$

3) $h(5) = f(g(5))-5$
$= f(3)-5$
$h(5) = 2$

4) By IVT, there is a value c for $1<c<5$ where $h(c)=-2$ b/c $-4<-2<2$.

---

10. Given $f(x) = \frac{x}{x-3}$ on the interval $[-2, 2]$. Determine if the IVT applies. State why or why not. Then, find the value of c such that $f(c) = \frac{1}{3}$.

1) f(x) is continuous on [-2, 2]

2) $f(-2) = 2/5$ and $f(2) = -2$

3) $f(c) = 1/3$ since $f(2) < 1/3 < f(-2)$ by IVT.

$\frac{c}{c-3} = \frac{1}{3}$
$3c = c-3$
$2c = -3$
$c = -3/2$

$\therefore$ $c = \frac{-3}{2}$ is value $x=c$ for $f(\frac{-3}{2}) = \frac{1}{3}$ for f(x) on $-2<x<2$.

---

11. Show that there is a value c with $0 < c < 2$ such that $x^2 + \cos \pi x = 4$. Then, use a graphing utility to find the approximate value of c.

1) f(x) is continuous on reals.

2) $f(0)=1$ and $f(2)=5$

3) $f(c)=4$ since by IVT, $1<4<5$ on $0<x<2$

* It can be shown by graphing that $c \approx 1.791$.

12. Does the IVT apply to the function $h(x) = \frac{x^2+x}{x-2}$ on the interval [2.5, 5]? If so, find the value of c guaranteed to exist, such that $h(c) = 12$.

1) $h(x)$ is continuous on $2.5<x<5$

2) $h(2.5)= 17.5$ and $h(5)=10$

3) Since $10<12<17.5$, there exists a value $x=c$ such that $h(c)= 12$ on $2.5<x<5$ by IVT.

4) $\frac{c^2+c}{c-2}=12$

$c^2+c=12c-24$

$(c-8)(c-3)=0$

$c\neq {}^-8, c=3$

∴ $c=3$ is valued guaranteed by IVT where $h(3)=12$ on $2.5\leq x\leq 5$.

---

13. Does the IVT apply to the function $f(x) = -\left(\frac{1}{2}\right)^{3-x} - 3$ on the interval [2,5] for $f(c) = -4$?

1) $f(x)$ is continuous on reals

2) $f(2) = {}^-\left(\frac{1}{2}\right)-3$ $f(5)= {}^-\left(\frac{1}{2}\right)^{-2}-3$

$f(2)= -3.5$ $f(5)= {}^-7$

3) By IVT $-7<{}^-4<-3.5$ on $2.5\leq x\leq 5$, so there exists a value, c, such that $f(c)={}^-4$

4) $-\left(\frac{1}{2}\right)^{3-c}-3=-4$

$-\left(\frac{1}{2}\right)^{3-c}=-1$

$\left(\frac{1}{2}\right)^{3-c}=1$

$\log_{1/2}\left(\frac{1}{2}\right)^{3-c}=\log_{1/2}1$

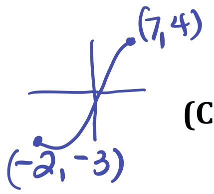
$3-c=0$

$\underline{\underline{c=3}}$

__C__ **14.** Let f be a continuous function on the closed interval $[-2, 7]$. If $f(-2) = -3$ and $f(7) = 4$, then the Intermediate Value Theorem guarantees that

(A) $f(0) < 0$ False

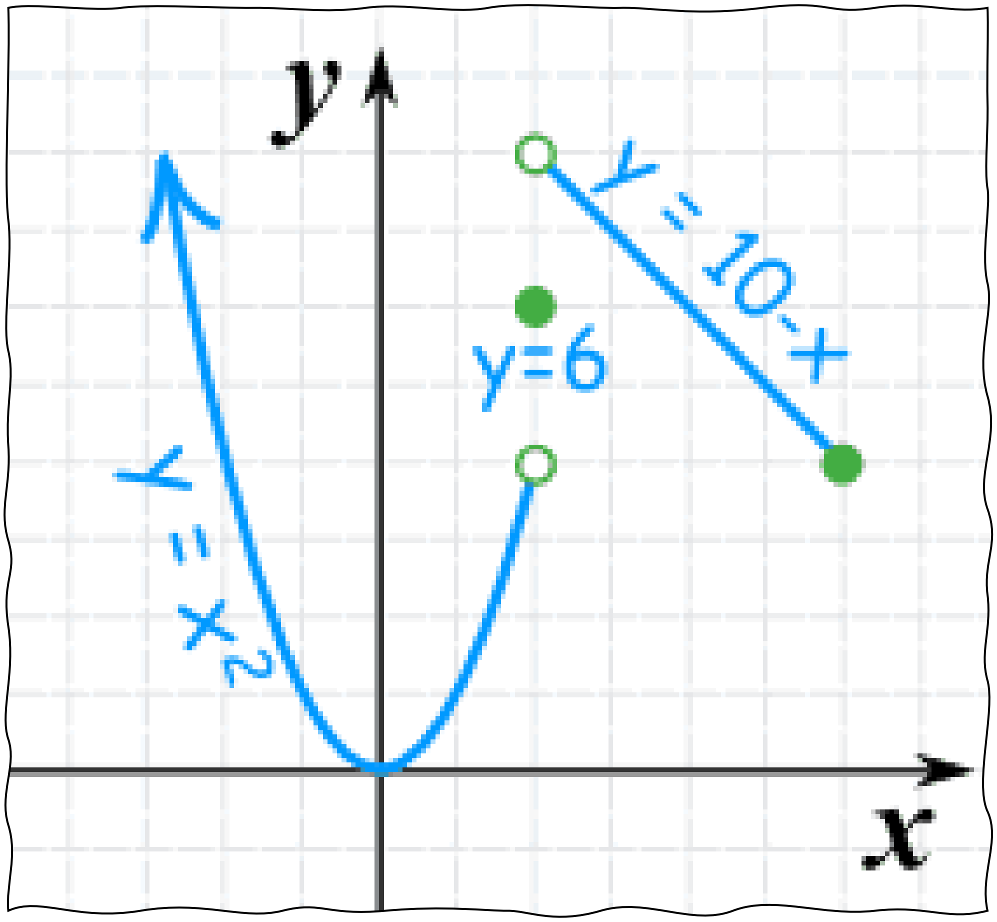
(B) $-3 \leq f(x) \leq 4$ for all x between -2 and 7. False



(C) $f(c) = 1$ for at least one c between -2 and 7 True

(D) $f(c) = 0$ for at least one c between -3 and 4 False

y-values



UNIT 1:

LIMITS & CONTINUITY

Lesson 1: The Concept of Instantaneous Rate of Change

Topic 1.1: Introducing Calculus: Can Change Occur at an Instant?

As students of life, and algebra, you have studied many rates of change for different situations: how fast we drive, how our savings grow over time, the rate that the world population changes, our growth rate every year, unemployment rates, and so much more. In each of these studies, we can express one variable in terms of another, that is, as a function, $y = f(x)$.

Velocity

Consider an object that travels in a straight line. The **average velocity** over some given time interval can be defined as the ratio of the change in position (**displacement** or **net change)** to the elapsed time period:

$$\text{Average Velocity} = \frac{\Delta \text{ position}}{\Delta \text{ time}} = \frac{\Delta d}{\Delta t}$$

However, when we think of velocity, we usually mean **instantaneous velocity**, which indicates the speed and direction of some object at a particular moment, which could be faster or slower than the average velocity.

Can you explain why we can't define instantaneous velocity as a ratio?

The time interval is zero, division by zero.

How can you estimate an instantaneous velocity?

By making the time interval very small.

Let's generalize that big idea with this principle:

Average velocity over a very small time interval is very close to instantaneous velocity.

EX #1: If you climb to the top of the Tower of Pisa and drop a coin to the ground, estimate the instantaneous velocity at $t = 0.7$ seconds. Use Galileo's formula $s(t) = 16t^2$ to compute the average velocity, in feet/second , over the time intervals listed in the table below.

Time Interval	Average Velocity
$[0.7, 0.71]$	22.56
$[0.7, 0.705]$	22.48
$[0.7, 0.7001]$	22.4016
$[0.7, 0.70005]$	22.4008
$[0.7, 0.700001]$	22.40016

What is your estimate for the ***Instantaneous Rate of Change*** (IROC) at $t = 0.7$ second?

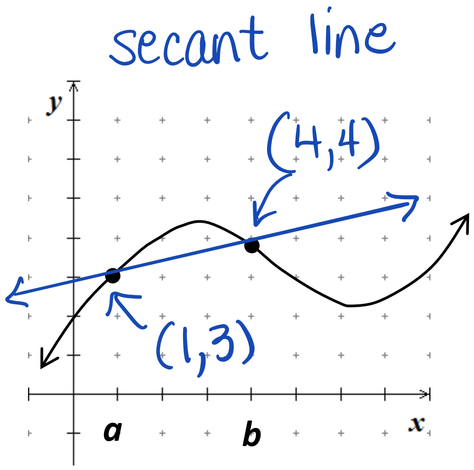
The coin is falling at a rate of about 22.400 ft/sec at $t = 0.7$ second.

Photo: Jean Adams

THE STUDY OF RATE OF CHANGE IS FUNDAMENTAL TO THE STUDY OF CALCULUS.

In our first example, we allowed the time interval to shrink to zero, so can say that *the average velocity* ***converges*** *to the instantaneous velocity* or that the *instantaneous velocity is the* ***limit*** *of the average velocity.*

Graphical Representation of Velocity



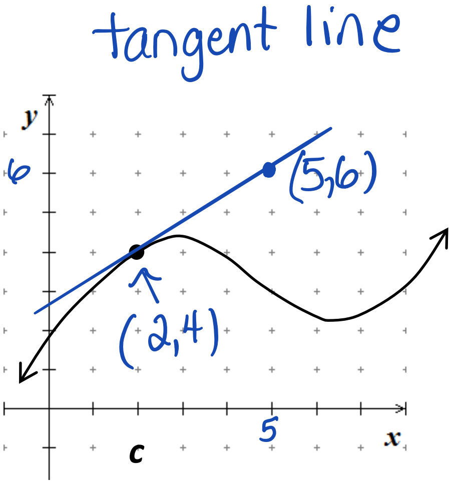
Average Speed (AROC) is defined by the total distance traveled, *d*, divided by the elapsed time period, *t*.

Graphically, the average speed is the slope of the secant line.

Analytically, average speed AROC $= \frac{\Delta d}{\Delta t}$

Sketch the secant line through *a* and *b*. *Estimate the average velocity.*

$$AROC = m_{sec} = \frac{4-3}{4-1} = \underline{\underline{\frac{1}{3}}}$$



Instantaneous Speed (IROC) is the speed at a specific instant in time, where $t = t_0$ is the limit of the average rates of change.

Graphically, the instantaneous speed is the slope of the tangent line at a specific point.

Analytically, we need ***calculus*** to determine this instantaneous speed.

Sketch the tangent line through point *c*. *Can you estimate the instantaneous velocity at* $x = c$?

Estimated IROC at (2,4) is about $\frac{2}{3}$

EX #2: Spin Class Heart Rate and Exercise

Have you ever taken a Spin Class at the gym? Then you know that after the class is over, your rapid heart rate decreases as time passes. Let $H(t)$ represent your heart rate. Write an expression for the rate of change of your heart rate over the period from $t = 0$ seconds to $t = 90$ seconds, after you have stopped exercising.

$$\frac{\Delta H}{\Delta t} = \frac{H(90) - H(0)}{90 - 0} = \frac{H(90) - H(0)}{90}$$

EX #3: Roller Coaster

A **hypercoaster** is a special circuit roller coaster with a height measuring greater than 200 feet. If a new hypercoaster is under design to have its largest drop on the circuit modeled by the equation below, where $d(t)$ is measured in feet, and time t, is measured in seconds.

$$d(t) = 1.87t^3 - 20.64t^2 + 112.79t + 132.56$$

A. Find the average speed (velocity) of the hypercoaster from 2 seconds to 8 seconds.

$$AROC = \frac{d(8)-d(2)}{8-2} = \frac{671.36-290.54}{6}$$

AROC on $2 \le t \le 8$ seconds is 63.47 ft/sec.

B. What is the average speed (velocity) of the hypercoaster from 4 seconds to 6 seconds?

$$AROC = \frac{d(6)-d(4)}{6-4} = \frac{470.18-373.16}{2}$$

AROC on $4 \le t \le 6$ seconds is 48.51 ft/sec.

C. Estimate the instantaneous speed (velocity) of the hypercoaster at exactly 5 seconds.

Time Interval	Average Velocity
$[5, 5.1]$	47.3997 ft/sec
$[5, 5.01]$	46.7142 ft/sec
$[5, 5.001]$	46.6474 ft/sec

The instantaneous velocity of the hypercoaster at $t = 5$ seconds is about 46.647 ft/sec.

D. Explain the meaning of the instantaneous speed (velocity)in the context of the question at $t = 5$ seconds.

The rate of change in the drop of the hypercoaster at $t=5$ seconds is about 46.647 feet per second.

EX #4: Airport Departures

The traffic pattern of departing flights at Orlando International Airport on a fall afternoon can be modeled by the function V defined by $V(t) = 92 - 15\sin\left(\frac{t}{3}\right)$ where $V(t)$ is measured in vehicles and t is measured in minutes $0 \leq t \leq 30$.

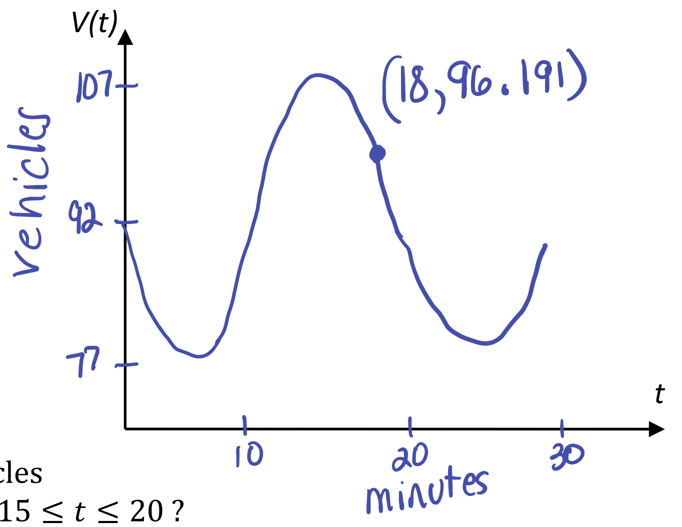
A. Find $V(t)$ for $t = 18$.

$V(18) = 92 - 15\sin\left(\frac{18}{3}\right)$

$V(18) \approx 96.1912$

about 96 vehicles

B. Make a sketch of the graph of time *vs.* number of vehicles. Be sure your graph has a scale.



C. What is the average rate of change of the vehicles at the departing flights over the time interval $15 \leq t \leq 20$? Indicate units of measure.

$$\text{AROC} = \frac{V(20) - V(15)}{20 - 15} \approx \frac{86.3877 - 106.3838}{5} \approx -3.9992 \frac{\text{veh}}{\text{min}}$$

Average rate of change on [15, 20] minutes is decreasing at a rate of about 4 vehicles per minute.

D. Estimate the instantaneous rate of change of vehicles at $t = 18$ by finding the average rates from $t = 18$ to $t = 18.1$, $t = 18$ to $t = 18.01$, and $t = 18$ to $t = 18.001$.

Time Interval	Average Velocity
[18, 18.1]	$-4.823 \frac{\text{veh}}{\text{min}}$
[18, 18.01]	$-4.803 \frac{\text{veh.}}{\text{min}}$
[18, 18.001]	$-4.801 \frac{\text{veh}}{\text{min}}$

The instantaneous rate of change at t=18 minutes is decreasing at a rate of about 4.8 vehicles per minute.

E. Why can't the instantaneous rate of change of traffic in the departure lane with respect to time be calculated using the method in part D?

Division by zero would occur.

Lesson 2: Understanding Limits Graphically and Numerically

Topic 1.2: Defining Limits and Using Limit Notation

Limits are the "backbone" of understanding that connect algebra and geometry to the mathematics of calculus. In basic terms, a limit is just a statement that tells you what height a function *INTENDS TO REACH* as you get close to a specific x-value. Recall from Pre-Calculus that you evaluated three types of limits. Complete the table below:

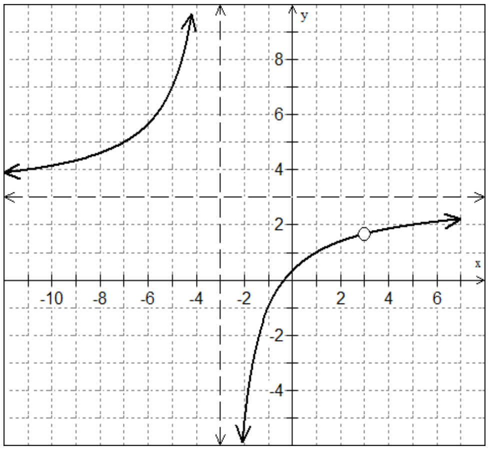
PROPER LIMIT NOTATIONS		
TYPE OF LIMIT	**PROPER NOTATION**	**VERBALLY:**
Right-hand limit	$\lim_{x \to c^+} f(x)$	limit as x approaches c from the right...
Left-hand limit	$\lim_{x \to c^-} f(x)$	limit as x approaches c from the left...
General limit	$\lim_{x \to c} f(x)$	limit as x approaches c...

Let's begin our discussion of limits by analyzing a rational function and examining the graph.

EX #1: Use the equation for $f(x)$ and the graph of the function to analyze completely.

$$f(x) = \frac{3x^2 - 8x - 3}{x^2 - 9} = \frac{(3x+1)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}}$$

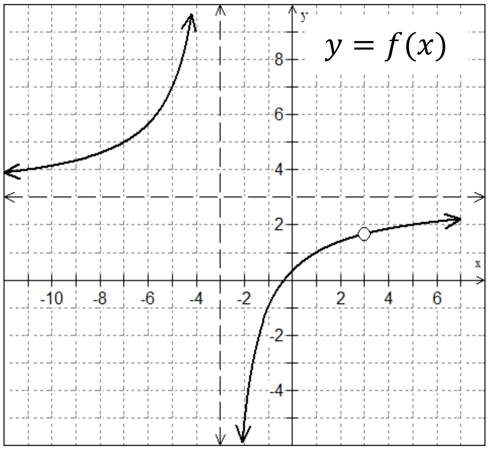
Factor and Simplify	$f(x) = \frac{3x+1}{x+3}$
Coordinates of Hole	$(3, 5/3)$
Domain	$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Range	$(-\infty, 5/3) \cup (5/3, 3) \cup (3, \infty)$
Vertical Asymptote	$x = -3$
Horizontal Asymptote	$y = 3$



Understanding Limit Notation

Let's revisit the notation we learned in Pre-Calculus. We need to convert these old methods of explaining extreme behaviors into limit notation for use in calculus.

EX #2: Use the graph to complete the table below.



Pre-Calculus Notation vs. Calculus Notation

As $x \to -\infty$, the graph of $f(x) \to$ 3 $\lim_{x\to-\infty} f(x) = 3$	As $x \to \infty$, the graph of $f(x) \to$ 3 $\lim_{x\to\infty} f(x) = 3$
As $x \to -3$ from the right, the graph of $f(x) \to$ $-\infty$ $\lim_{x\to -3^+} f(x) = -\infty$	As $x \to 3$ from the left, the graph of $f(x) \to$ 5/3 $\lim_{x\to 3^-} f(x) = 5/3$
As $x \to -3$ from the left, the graph of $f(x) \to$ ∞ $\lim_{x\to -3^-} f(x) = \infty$	As $x \to 3$ from the right, the graph of $f(x) \to$ 5/3 $\lim_{x\to 3^+} f(x) = 5/3$

Can you explain what the value of a limit represents in terms of the graph?

The limit is the "y-value" that the graph is approaching from both the left side and the right side of the "target value" of x=c. The limit can occur at a "hole," a removable discontinuity.

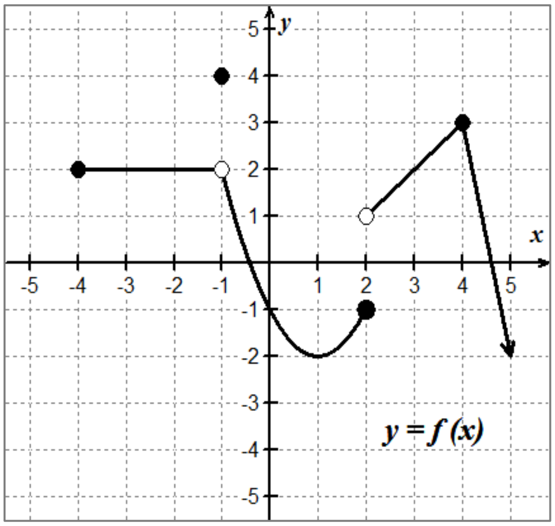
Informally, a **limit is a** *y*-value which a function approaches as *x* approaches some value.

$\lim_{x\to c} f(x) = L$ means as x approaches c, $f(x)$ approaches the y-value of L.

Topic 1.3: Estimating Limit Values from Graphs

Consider the function shown at right.

Say you want to find $\lim_{x \to 4^+} f(x)$, the positive sign in the limit notation indicates a right-hand limit. If you think of the function as a highway and imagine you are traveling along the graph of $f(x)$ toward $x = 4$ **FROM THE RIGHT, NOT TO THE RIGHT**, and you stop at the vertical line $x = 4$, the y-value where you stop is 3. Therefore, $\lim_{x \to 4^+} f(x) = 3$.



EX #3: Use the graph above to evaluate each of the following limits:

A. $f(2)$ $= -1$	**B.** $f(-1)$ $= 4$
C. $\lim_{x \to 4^-} f(x) = 3$	**D.** $\lim_{x \to 2^+} f(x) = 1$
E. $\lim_{x \to 2^-} f(x) = -1$	**F.** $\lim_{x \to -1^+} f(x) = 2$
G. $\lim_{x \to -1^-} f(x) = 2$	**H.** $\lim_{x \to -4^+} f(x) = 2$
I. $\lim_{x \to -4^-} f(x)$ d.n.e.	**J.** $\lim_{x \to -1} f(x) = 2$
K. $\lim_{x \to 2} f(x)$ d.n.e.	**L.** $\lim_{x \to 5} f(x) = -2$
M. $\lim_{x \to 0} f(x) = -1$	**N.** $\lim_{x \to 1} f(x) = -2$

Think about this!

If we think of the function as a highway, then the point at $(2, -1)$ could be considered the end of the road, while the point at $(-1, 2)$ is more like a "pothole." How would you describe the points located at

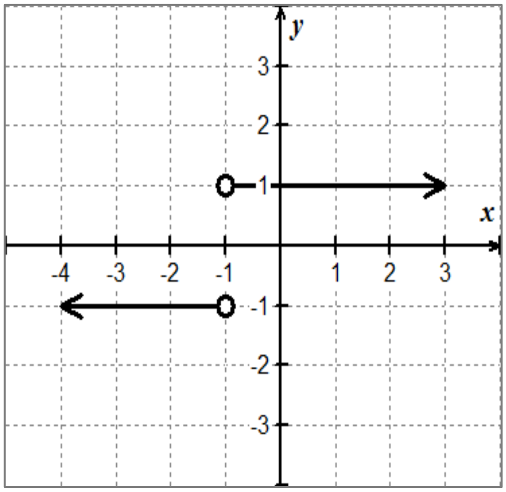
$(2, 1)$: dead end without a barrier

$(4, 3)$: bump in the road

Hopefully, this analogy gives you a visual reference for understanding limits from a graphical approach. Let's get a little more formal with our definition now.

EX #4: Limits can fail to exist in three situations:

CASE 1: limits differ on left and right



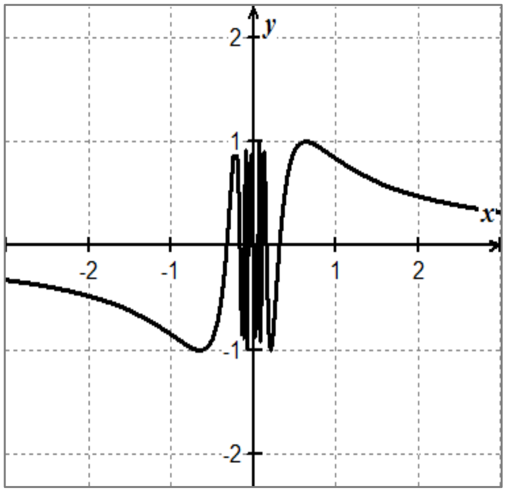
Justify why the limit does not exist at $x = -1$ for $f(x) = \frac{|x+1|}{x+1}$

$\lim_{x\to -1^+} f(x) = 1$

$\lim_{x\to -1^-} f(x) = -1$

$\therefore \lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$

CASE 2: Oscillating behavior



Justify why the limit does not exist at $x = 0$ for $f(x) = \sin\left(\frac{1}{x}\right)$

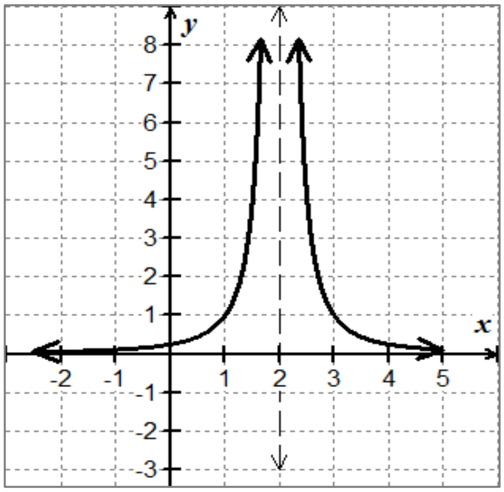
By inspection, look at $x = 0.1$

$\lim_{x\to 0^+} f(x) < 0$ while at $x = -0.1$

$\lim_{x\to 0^-} f(x) > 0$

$\therefore$ limit does not exist

CASE 3: unbounded behavior



Justify why the limit does not exist at $x = 2$ for $f(x) = \frac{1}{(x-2)^2}$

$\lim_{x\to 2^+} f(x) = \infty$ $\lim_{x\to 2^-} f(x) = \infty$

$\therefore \lim_{x\to 2} f(x) = \infty$

Note: by definition, ∞ is not a real number, this describes the behavior of f(x)

Topic 1.4: Estimating Limit Values from Tables

EX #5: Now, consider the function $f(x) = \frac{x-3}{x^2+2x-15}$. Complete the table below to find the limit as $x \to 3$.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	0.1265	0.1251	0.1250	undefined	0.1249	0.1248	0.1234

Based on your analysis, what are the values of each of the limits below?

$\lim_{x \to 3^-} f(x) =$ 0.125	$\lim_{x \to 3^+} f(x) =$ 0.125	$\lim_{x \to 3} f(x) =$ 0.125

By evaluating certain types of functions at a particular value, we may not necessarily have a sufficient understanding of the function's behavior at a specific point. This is especially true for rational functions that contain discontinuities.

This is why the idea of a ***limit is so important!*** Rather than just going directly to some x-value using direct substitution , we can **approach a point from either side** to get some sense of behavior in the **neighborhood**.

Look at the photo of the arch support construction for the Hoover Dam Bypass Bridge (2009). We can tell where the missing section is going to be. The actual height at that point would be the limit. With limits, we can discuss behavior at a point whether the point exists or not. With limits, we are looking at the y-value the graph approaches, not the actual y-value at that point.

When finding limits, ask yourself, "What is happening to y as x gets close to a certain number?" You are finding the ***y*-value** for which the function is approaching as x approaches c.

LIMIT EXISTENCE THEOREM:

$$\lim_{x \to c} f(x) \text{ exists if and only if } \lim_{x \to c-} f(x) = \lim_{x \to c^+} f(x) = L$$

where L is a real number.

Verbally: The limit as x approaches c on $f(x)$ will exist if and only if the limit as x approaches c from the left is equal to the limit as x approaches c from the right.

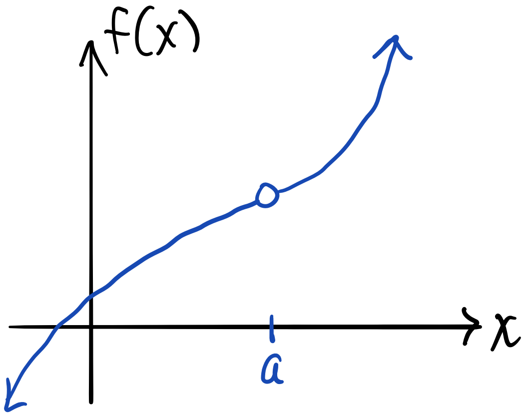
EX #6: YOU OWN IT! In the box below, complete the sentence in your own words.

In order for the GENERAL LIMIT to exist, the function:

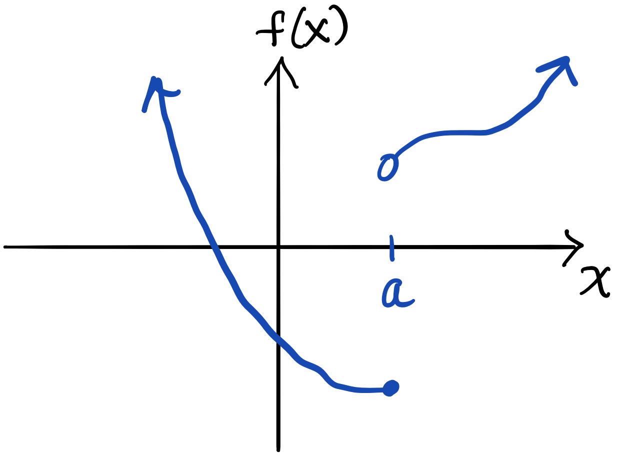
limit must approach the same y-value from the left-hand side and the right-hand side. However, the function value may or may not be defined here.

EX #7: Sketch a graph to satisfy each set of conditions.

1. $f(a)$ is undefined
2. $x = a$ is a point discontinuity
3. $\lim_{x \to a} f(x)$ exists



1. $\lim_{x \to a} f(x)$ DNE
2. $x = a$ is a jump discontinuity
3. $f(a)$ is defined



Answers may vary.

Lesson 3: Properties of Limits

Topic 1.5: Determining Limits Using Algebraic Properties of Limits

There will be times when you need to consider limits of functions symbolically, such as, $f(x)$, $g(x)$, and $h(x)$. When you are given combinations, compositions, and other modifications of these nonspecific functions, there are some important properties of limits that will prove essential to these general function types.

LIMIT PROPERTIES

Let b and c be real numbers, let n be a positive integer, and let f and g be functions where $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both exists, then

1. Constant Rule:

$$\lim_{x\to c} k = k$$

2. Identity Rule:

$$\lim_{x\to c} x = c$$

3. Coefficient Rule:

$$\lim_{x\to c}[b \cdot\ f(x)] = b \cdot \lim_{x\to c} f(x)$$

4. Sum or Difference Rule:

$$\lim_{x\to c}[f(x) \pm\ g(x)] = \lim_{x\to c} f(x) \pm\ \lim_{x\to c} g(x)$$

5. Product Rule:

$$\lim_{x\to c}[f(x) \cdot\ g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

6. Quotient Rule:

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}\ ; \lim_{x\to c} g(x) \neq 0$$

7. Power Rule:

$$\lim_{x\to c}[f(x)]^n = \left(\lim_{x\to c} f(x)\right)^n$$ **, n is rational**

8. Composite Function Rule:

$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)$$

9. Root Rule:

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$ if the root on the right side exists.

Being skilled at reciting these rules in English will help you truly understand them. For example, The Sum or Difference Rule can be stated as "*The limit of a sum (difference) is the sum (difference) of the limits.* Can you give a description for each property in your own words?

EX#1: Use the properties of limits to find each of the following limits. If a limit does not exist, state why.

A. $\lim_{x\to 3^+}[f(x)+g(x)]$

$$\lim_{x\to 3^+} f(x) + \lim_{x\to 3^+} g(x)$$

$$6 + 8 = \underline{\underline{14}}$$

B. $\lim_{x\to 5}[3f(x)-2g(x)]$

$$3\lim_{x\to 5} f(x) - 2\lim_{x\to 5} g(x)$$

$$3(2) - 2(4) = \underline{\underline{-2}}$$

C. $\lim_{x\to -5^-}[f(x)-g(x)]$

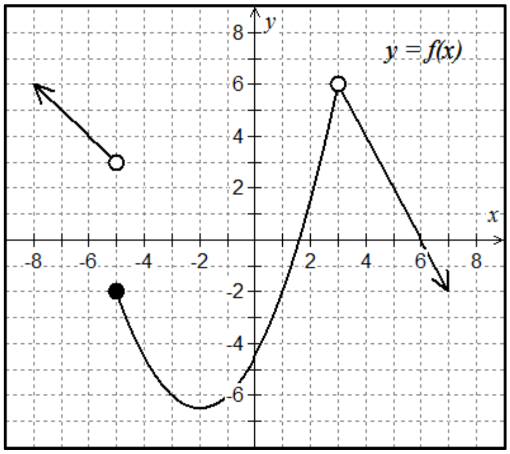
$$\lim_{x\to -5^-} f(x) - \lim_{x\to -5^-} g(x)$$

$$3 - 4 = \underline{\underline{-1}}$$

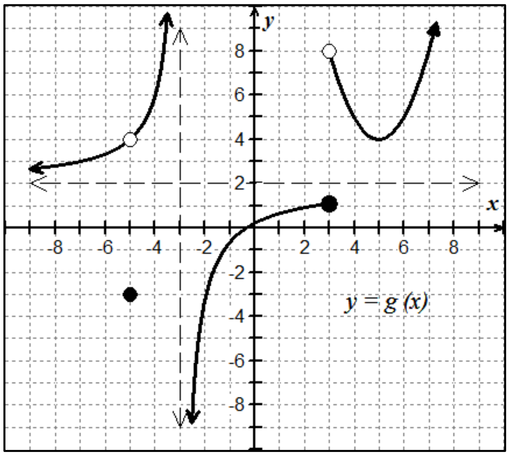
D. $\lim_{x\to -7}\frac{3f(x)}{g(x)}$

$$\frac{3\lim_{x\to -7} f(x)}{\lim_{x\to -7} g(x)} = \frac{3(5)}{3} = \underline{\underline{5}}$$

Graph of $f(x)$



Graph of $g(x)$



E. $\lim_{x\to 3} 4[f(x)\cdot g(x)]$

$$4\lim_{x\to 3} f(x)\cdot \lim_{x\to 3} g(x)$$

g is not continuous → $\underline{\underline{\text{DNE}}}$

F. $\lim_{x\to -3}[f(x)]^2$

$$\left[\lim_{x\to -3} f(x)\right]^2 = (-6)^2 = \underline{\underline{36}}$$

G. $\lim_{x\to 4}\sqrt{8\,g(x)}$

$$\left[\lim_{x\to 4} 8\cdot g(x)\right]^{1/2} = \sqrt{8(5)} = \underline{\underline{2\sqrt{10}}}$$

H. $\lim_{x\to -3}[f(x)+g(x)]$

$$\lim_{x\to -3} f(x) + \lim_{x\to -3} g(x)$$

g is not continuous → $\underline{\underline{\text{DNE}}}$

J. $\lim_{x\to 6}\frac{f(x)}{g(x)}$

$$\frac{\lim_{x\to 6} f(x)}{\lim_{x\to 6} g(x)} = \frac{0}{5} = \underline{\underline{0}}$$

K. $\lim_{x\to -2} f(g(x))$

$$f\left[\lim_{x\to -2} g(x)\right] = f(-3) = \underline{\underline{-6}}$$

EX #2: Find the limits if $\lim_{x\to c} f(x) = 3$, $\lim_{x\to c} g(x) = -2$, and $\lim_{x\to c} h(x) = 4$

A. $\lim_{x\to c}\sqrt{3\,h(x) - 2\,g(x)}$

$$\lim_{x\to c}\left[3\,h(x) - 2g(x)\right]^{1/2}$$

$$\left[3\lim_{x\to c} h(x) - 2\lim_{x\to c} g(x)\right]^{1/2}$$

$$\left[3(4) - 2(-2)\right]^{1/2}$$

$$\sqrt{16} = \underline{\underline{4}}$$

B. $\lim_{x\to c}\,[f(x)\cdot 5\,g(x)]$

$$\lim_{x\to c} f(x)\cdot 5\lim_{x\to c} g(x)$$

$$3(5)(-2)$$

$$-\underline{\underline{30}}$$

C. $\lim_{x\to c}[7 - g(x)]^2$

$$\left[\lim_{x\to c} 7 - \lim_{x\to c} g(x)\right]^2$$

$$\left[7-(-2)\right]^2$$

$$9^2$$

$$\underline{\underline{81}}$$

D. $\lim_{x\to c}\frac{2\,f(x) + 3\,h(x)}{h(x) - g(x)}$

$$\frac{2\lim_{x\to c} f(x) + 3\lim_{x\to c} h(x)}{\lim_{x\to c} h(x) - \lim_{x\to c} g(x)}$$

$$\frac{2(3) + 3(4)}{4-(-2)} = \frac{18}{6} = \underline{\underline{3}}$$

E. $\lim_{x\to c}[h(x)\cdot\ (f(x) + 6)]$

$$\left[\lim_{x\to c} h(x)\right]\left[\lim_{x\to c} f(x) + \lim_{x\to c} 6\right]$$

$$(4)\ [3+6]$$

$$4(9)$$

$$\underline{\underline{36}}$$

F. $\lim_{x\to c}\frac{f(x)^2}{4 - g(x)}$

$$\frac{\left[\lim_{x\to c} f(x)\right]^2}{\lim_{x\to c} 4 - \lim_{x\to c} g(x)} = \frac{3^2}{4-(-2)}$$

$$= \frac{9}{6} = \underline{\underline{\frac{3}{2}}}$$

EX #3: Given $\lim_{x\to c} f(x) = 8$ and $\lim_{x\to c} g(x) = -4$, then find the following:

$$\lim_{x\to c}\left\{2f(x)\cdot g(x) - \sqrt[3]{f(x)} + 2\,[g(x)]^2\right\}$$

$$= 2\lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x) - \left[\lim_{x\to c} f(x)\right]^{1/3} + 2\left[\lim_{x\to c} g(x)\right]^2$$

$$= 2(8)(-4) - 8^{1/3} + 2(-4)^2$$

$$= -64 - 2 + 32$$

$$= \underline{\underline{-34}}$$

EX #4: Find the limits of the following composition functions.

A. Given $f(x) = x^3 - x^2 + 4x - 1$ and $g(x) = -2x$, find $\lim\limits_{x\to 2} f(g(x))$.

$f(\lim\limits_{x\to 2} g(x)) = f(-2(2))$

$f(-4) = (-4)^3 - (-4)^2 + 4(-4) - 1$

$f(-4) = -64 - 16 - 16 - 1$

$\underline{\underline{f(-4) = -97}}$

B. Given $f(x) = x^3 - x^2 + 4x - 1$ and $g(x) = -2x$, find $\lim\limits_{x\to 2} g(f(x))$

$g(\lim\limits_{x\to 2} f(x))$

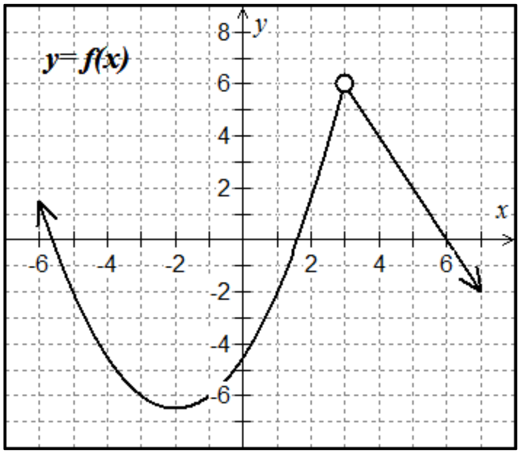
$g(2^3 - 2^2 + 4(2) - 1)$

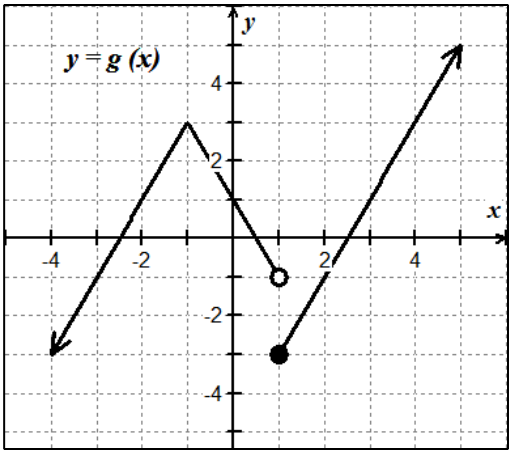
$g(11)$

$g(11) = -2(11)$

$\underline{\underline{g(11) = -22}}$

EX #5: Use the graphs of $f(x)$ and $g(x)$ below to find the limits.





A. $\lim\limits_{x\to 3} f(g(x))$

$f[\lim\limits_{x\to 3} g(x)]$

$= f(1)$

$= \underline{\underline{-2}}$

B. $\lim\limits_{x\to -1} g(f(x))$

$g[\lim\limits_{x\to -1} f(x)] = g(-6) = \underline{-7}$

$g(x) = 2x + 5$

$g(-6) = -12 + 5 = -7$

C. $\lim\limits_{x\to 1} f(g(x))$

$\lim\limits_{x\to 1} g(x)$ dne

$\begin{cases} x\to 1^- & g\to 1^+ \\ x\to 1^+ & g\to -3^+ \end{cases}$

$\lim\limits_{x\to -1^+} f(x) = -6$

$\lim\limits_{x\to -3^+} f(x) = -6$

$\therefore \lim\limits_{x\to 1} f(g(x)) = \underline{\underline{-6}}$

D. $\lim\limits_{x\to -3} g(f(x))$

$g[\lim\limits_{x\to -3} f(x)]$

$g(-6) = \underline{\underline{-7}}$

Lesson 4: Limits of Composite Functions

Topic 1.7: Selecting Procedures for Determining Limits
Topic 1.9: Connecting Multiple Representations of Limits

The limit of a composition is the composition of the limits, provided the outside function is continuous at the limit of the inside function. We often see this IF-THEN STATEMENT presented as follows:

$$f \text{ and } g \text{ are functions such that } \lim_{x\to c} g(x) = L \text{ and } \lim_{x\to L} f(x) = f(L) \quad \text{then}$$

$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right) = f(L)$$

or stated as,

$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right) \text{ provided } \lim_{x\to c} g(x) = L \text{ and } f(x) \text{ is continuous at } x = L$$

We should notice this IF-THEN statement requires certain conditions to be met. Problems occur when $f(x)$ is not continuous at $x = L$ or the existence of the limit of $g(x)$ at $x = a$. At issues of this sort, we can't apply the property. You might assume that the limit doesn't exist, but this may or may not be true. So, we will explore some alternative plan to find another approach.

In this case, let's apply the "*if and only if*" definition of a limit:

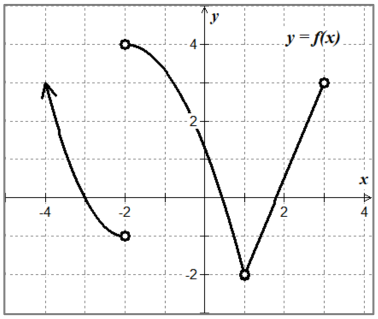
$$\lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x) \quad \text{then} \quad \lim_{x\to c} f(x) = L$$

EX #1: $\lim_{x\to 1} f\big(f(x)\big)$

Let $u = \lim_{x\to1} f(x) = -2^+$ from "above"

$\lim_{u\to -2^+} f(u) = 4$

$\therefore \underline{\lim_{x\to1} f(f(x)) = 4}$



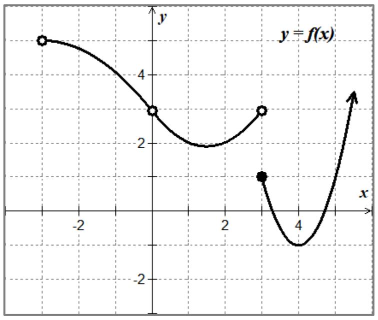
EX #2: $\lim_{x\to 0} f\big(f(x)\big)$

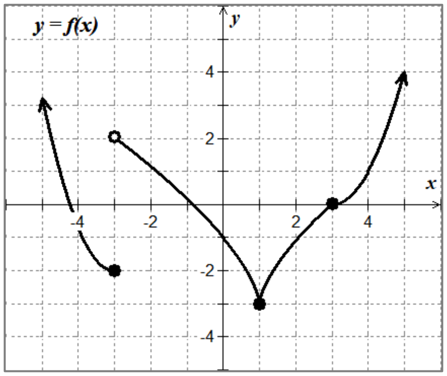
Let $u = \lim_{x\to0} f(x) = 3$ from above + below

$\lim_{u\to3^-} f(u) = 3$ and $\lim_{u\to3^+} f(u) = 1$

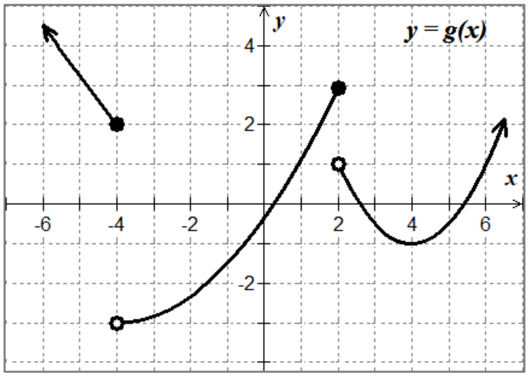
$3 \neq 1$

$\therefore \underline{\lim_{x\to0} f(f(x)) \text{ d.n.e}}$

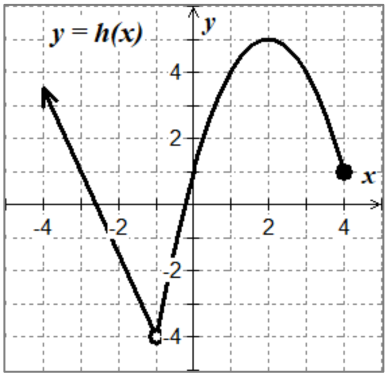




$y = f(x)$



$y = g(x)$



$y = h(x)$

EX #3: $\lim_{x\to1} f(f(x))$

Let $u = \lim_{x\to1} f(x) = -3^+$ from above

$\lim_{u\to-3^+} f(u) = 2$

$\therefore \lim_{x\to1} f(f(u)) = 2$

EX #4: $\lim_{x\to2} h(g(x))$

Let $u = \lim_{x\to2^-} g(x) = 3^-$ below and $u = \lim_{x\to2^+} g(x) = 1^-$ below

$$\left.\begin{array}{l}\lim_{u\to3^-} h(u) = 4 \\ \lim_{u\to1^-} h(u) = 4\end{array}\right\}\text{same}$$

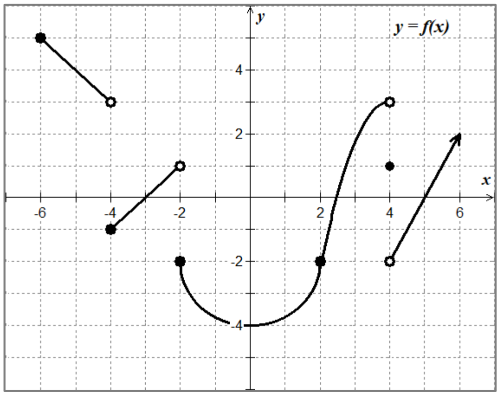
$\therefore \lim_{x\to2} h(g(x)) = 4$

EX #5: $\lim_{x\to-1} g(h(x))$

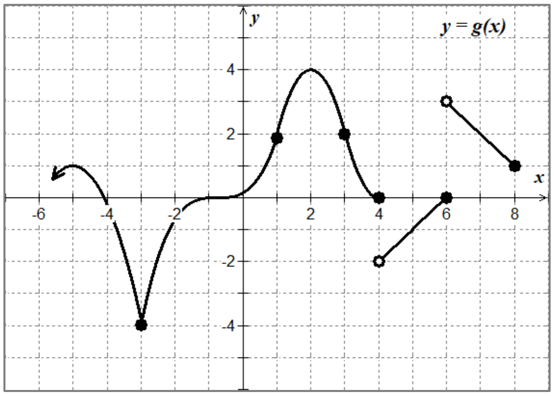
Let $u = \lim_{x\to-1} h(x) = -4^+$ above

$\lim_{u\to-4^+} g(u) = -3$

$\therefore \lim_{x\to-1} g(h(x)) = -3$



$y = f(x)$



$y = g(x)$

EX #6: $\lim_{x \to -3} f(g(x))$

Let $u = \lim_{x \to {}^-3} g(x) = -4^+$ above

$\lim_{u \to -4+} f(u) = {}^-1$

$\therefore \lim_{x \to {}^-3} f(g(x)) = {}^-1$

EX #7: $\lim_{x \to 2} f(g(x))$

Let $u = \lim_{x \to 2} g(x) = 4^-$ below

$\lim_{u \to 4^-} f(u) = 3$

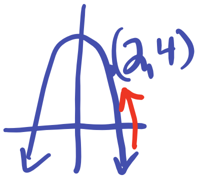
$\therefore \lim_{x \to 2} f(g(x)) = 3$

EX #8: $\lim_{x \to 2^+} f(8 - x^2)$

Let $u = 8 - x^2$

as $x \to 2^+$ $u \to 4^-$

from below



$\lim_{u \to 4^-} f(u) = 3$

$\therefore \lim_{x \to 2^+} f(8 - x^2) = 3$

EX #9: $\lim_{x \to -4^+} g(2 + f(x))$

Let $u = \lim_{x \to {}^-4^+} 2 + \lim_{x \to {}^-4^+} f(x)$

$(2) + (-1) = 1$

$\lim_{u \to 1} g(u) = 2$

$\therefore \lim_{x \to -4^+} g(2 + f(x)) = 2$

Lesson 5: Finding Limits by Analytic Methods

Topic 1.6: Determining Limits Using Algebraic Manipulation

If you only observe the graph of a function, it can be misleading when finding the limit of a function. In this lesson we will explore how to find limits using algebraic techniques and limit theorems.

You will learn to analyze limits by the following methods:

Methods to Analyze Limits:

1. Direct substitution.
2. Basic Limit Theorems
3. Factor, cancellation technique. Then go back to step 1.
4. The conjugate method, rationalize the numerator. Then, go back to step 2.
5. Special trig limits of $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ or $\lim\limits_{x\to 0}\frac{1-\cos x}{x} = 0$
6. L'Hospital's Rule (presented in Unit 3)

Substitution Theorem

If f is a polynomial function or rational function, then $\lim\limits_{x\to c} f(x) = f(c)$ provided that if f is a rational function the value of the denominator does not equal 0.

NOTE: ***Always try DIRECT SUBSTITUTION first.*** *If you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ the goal will be to simplify the expression using algebraic techniques and then try substitution again.*

EX #1: Find each of the following limits analytically using direct substitution.

A. $\lim\limits_{x\to 2}(3x^2 - 5x + 4)$

$3(2)^2 - 5(2) + 4 = \underline{\underline{6}}$

B. $\lim\limits_{x\to 2}\frac{x^3+1}{x+1}$

$\frac{2^3+1}{2+1} = \frac{9}{3} = \underline{\underline{3}}$

C. $\lim\limits_{x\to e}\frac{\ln x}{3x}$

$\frac{\ln e}{3e} = \underline{\underline{\frac{1}{3e}}}$

D. $\lim\limits_{x\to 4}\sqrt[3]{x+4}$

$\sqrt[3]{4+4} = \sqrt[3]{8} = \underline{\underline{2}}$

E. $\lim\limits_{\theta\to\frac{\pi}{6}}\sin 2\theta$

$\sin 2\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \underline{\underline{\frac{\sqrt{3}}{2}}}$

F. $\lim\limits_{x\to 5}\log_3(x+4)$

$\log_3(5+4) = \log_3 9 = \underline{\underline{2}}$

Topic 1.10: Exploring Types of Discontinuities

What is the process for finding discontinuities of a rational function from Pre-calculus?

1) Factor numerator and denominator, simplify
2) A point discontinuity occurs when factors are removed (cancel) when $(x-a)=0$.
3) A non-removable discontinuity occurs if factors don't cancel in the denominator, when $(x-a)=0$

You can perform the same algebraic analysis to find the limit of the removable, or point discontinuities and the non-removable, or infinite discontinuities using what we will call the **Factoring Method or Cancellation Technique**.

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-6x+8}$$

$$\lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{(x-4)\cancel{(x-2)}} \Rightarrow \lim_{x\to 2} \frac{x+3}{x-4}$$

$$= \frac{2+3}{2-4} = -\frac{5}{2}$$

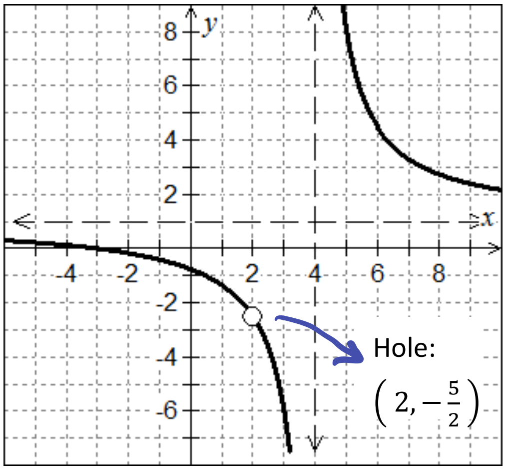
Topic 1.9: Connecting Multiple Representations of Limits

Graphically looking at the function, we see that just because it is undefined at a specific x-value doesn't mean that we can't find the limit. *Remember the Hoover Dam construction example.* Use the graph of the function to determine the value of each limit below.

$\lim_{x\to 2} \frac{x^2+x-6}{x^2-6x+8} =$ $-\frac{5}{2}$

$\lim_{x\to 4^+} \frac{x^2+x-6}{x^2-6x+8} =$ ∞

$\lim_{x\to 4^-} \frac{x^2+x-6}{x^2-6x+8} =$ $-\infty$



Algebraically Finding Limits of Functions at Undefined Values

Consider what happens when you try to evaluate this limit using direct substitution.

direct substitution leads to $\frac{0}{0}$. [indeterminate form]

REMEMBER: When you get indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ your goal is to algebraically manipulate the expression in an effort to remove the point(s) of discontinuity. Then, try direct substitution again. Let's explore a few techniques.

Finding One-Sided Limits

Recall, in our example there is a vertical asymptote at $x = 4$.
How can we determine the behavior of a function at this value of x?

$$\lim_{x\to 4^+} \frac{x^2+x-6}{x^2-6x+8} = \lim_{x\to 4^+} \frac{x+3}{x-4}$$

As $x \to 4^+$, pick a value to the right of 4, then analyze the simplified function:

$$\frac{(4.1)+3}{(4.1)-4} = \frac{7.1}{0.1} \qquad \frac{(+)}{(+)}$$

As $x \to 4^+$, $y \to \infty$

$$\lim_{x\to 4^+} f(x) = \infty$$

$$\lim_{x\to 4^-} \frac{x^2+x-6}{x^2-6x+8} = \lim_{x\to 4^-} \frac{x+3}{x-4}$$

As $x \to 4^-$, pick a value to the left of 4, then analyze the simplified function:

$$\frac{(3.9)+3}{(3.9)-4} = \frac{6.9}{-0.1} \qquad \frac{(+)}{(-)}$$

As $x \to 4^-$, $y \to -\infty$

$$\lim_{x\to 4^-} f(x) = -\infty$$

EX #2: Finding limits analytically of piecewise functions. Show your algebraic steps.

A. $\lim_{x\to 1} g(x)$ given, $g(x) = \begin{cases} x^3+1, & x > 1 \\ x+1, & x \le 1 \end{cases}$

$$\lim_{x\to 1^+} g(x) = (1)^3+1 = 2$$

$$\lim_{x\to 1^-} g(x) = (1)+1 = 2$$

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x)$$

$$\therefore \underline{\lim_{x\to 1} g(x) = 2}$$

B. $\lim_{x\to 2} h(x)$ given, $h(x) = \begin{cases} x^2-4x+7, & x \le 2 \\ -x^2+4x-1, & x > 2 \end{cases}$

$$\lim_{x\to 2^-} h(x) = 2^2-4(2)+7 = 3$$

$$\lim_{x\to 2^+} h(x) = -(2)^2+4(2)-1 = 3$$

$$\lim_{x\to 2^-} h(x) = \lim_{x\to 2^+} h(x)$$

$$\therefore \underline{\lim_{x\to 2} h(x) = 3}$$

Topic 1.7: Selecting Procedures for Determining Limits

EX #3: The Factoring or Cancellation Technique

A. $\lim\limits_{x\to -4} \frac{2x^2+7x-4}{x^2-x-20}$

$$= \frac{2(-4)^2+7(-4)-4}{(-4)^2-(-4)-20}$$

$$\frac{0}{0}$$

$$\lim_{x\to -4} \frac{(2x-1)\cancel{(x+4)}}{(x-5)\cancel{(x+4)}}$$

$$= \frac{2(-4)-1}{-4-5} = \frac{-9}{-9} = \underline{\underline{1}}$$

B. $\lim\limits_{x\to 3} \frac{x^3-27}{x^2+x-12}$

$$\frac{3^3-27}{3^2+3-12}$$

$$\frac{0}{0}$$

$$\lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}(x+4)}$$

$$= \frac{(3)^2+3(3)+9}{3+4} = \underline{\underline{\frac{27}{7}}}$$

EX #4: The LCM/LCM Method for Complex Fractions

A. $\lim\limits_{x\to 0} \frac{x}{\frac{1}{4}+\frac{1}{x-4}}$ direct substitute is 0/0

LCM : 4(x-4)

$$\lim_{x\to 0}\left[\frac{x}{\frac{1}{4}+\frac{1}{x-4}}\right]\left[\frac{4(x-4)}{4(x-4)}\right]$$

$$\lim_{x\to 0} \frac{4x\cancel{(x-4)}}{\cancel{(x-4)}+4}$$

$$\lim_{x\to 0} \frac{4\cancel{x}(x-4)}{\cancel{x}}$$

$$4(0-4) = \underline{\underline{-16}}$$

B. $\lim\limits_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}$ direct substitute is 0/0

LCM: 2(2+x)

$$\lim_{x\to 0}\left[\frac{\frac{1}{2+x}-\frac{1}{2}}{x}\right]\left[\frac{2(2+x)}{2(2+x)}\right]$$

$$\lim_{x\to 0} \frac{\cancel{2}-(\cancel{2}+x)}{2x(2+x)}$$

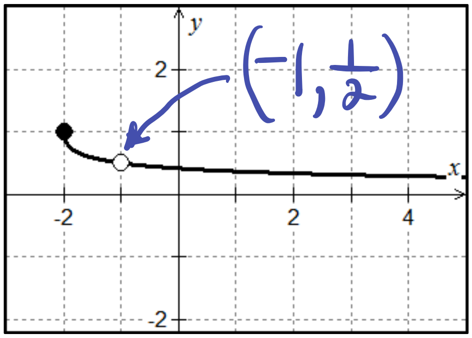
$$\lim_{x\to 0} \frac{-1\cancel{x}}{2\cancel{x}(2+x)}$$

$$\frac{-1}{2(2+0)} = \underline{\underline{-\frac{1}{4}}}$$

EX #5: The Rationalization Technique or Conjugate Method

The technique of rationalization can be used to find the limit when there is a radical in the numerator or denominator.

A. The graph of $g(x) = \frac{\sqrt{x+2}-1}{x+1}$ is shown at right.



$$\lim_{x\to -1}\left[\frac{\sqrt{x+2}-1}{x+1}\right]\left[\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}\right]$$

$$\lim_{x\to -1}\frac{\cancel{(x+2-1)}}{\cancel{(x+1)}(\sqrt{x+2}+1)}$$

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1} = \frac{1}{\sqrt{-1+2}+1} = \underline{\underline{\frac{1}{2}}}$$

$$\therefore \underline{\underline{\lim_{x\to -1} g(x) = \frac{1}{2}}}$$

B. $\lim_{x\to 5}\left(\frac{x-5}{3-\sqrt{x+4}}\right)\left(\frac{3+\sqrt{x+4}}{3+\sqrt{x+4}}\right)$

$$\lim_{x\to 5}\frac{(x-5)(3+\sqrt{x+4})}{9-(x+4)} = \lim_{x\to 5}\frac{\overset{-1}{\cancel{(x-5)}}(3+\sqrt{x+4})}{\cancel{(5-x)}}$$

$$= \lim_{x\to 5}(-1)(3+\sqrt{x+4}) = -1(3+\sqrt{9}) = \underline{\underline{-6}}$$

EX #6: Find each of the following limits analytically. Show your algebraic steps.

A. $\lim_{x\to 3}(5x+1)^{\frac{2}{3}}$

$$\left(\sqrt[3]{5(3)+1}\right)^2$$

$$\left(\sqrt[3]{8\cdot 2}\right)^2$$

$$\left(2\sqrt[3]{2}\right)^2$$

$$\underline{\underline{4\sqrt[3]{4}}}$$

B. $\lim_{x\to 3}\left(\frac{\sqrt{x+1}-2}{x-3}\right)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right)$

$$\lim_{x\to 3}\frac{\cancel{(x+1-4)}\cdot 1}{\cancel{(x-3)}(\sqrt{x+1}+2)}$$

$$\frac{1}{\sqrt{3+1}+2} = \underline{\underline{\frac{1}{4}}}$$

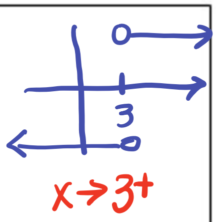
C. $\lim_{x\to -3^-} \frac{x^2|2x+6|}{4x+12}$ rewrite as piecewise

$\frac{\cancel{2}x^2\cancel{(x+3)}}{2\cancel{4}\cancel{(x+3)}}$ if $x \ge -3$

$\frac{-\cancel{2}x^2\cancel{(x+3)}}{2\cancel{4}\cancel{(x+3)}}$ if $x < -3$

$\therefore \lim_{x\to -3^-} \frac{-x^2}{2} = \frac{-9}{2}$

D. $\lim_{x\to 3^+} \frac{4x-12}{\sqrt{x^2-6x+9}}$



$\lim_{x\to 3^+} \frac{4(x-3)}{\sqrt{(x-3)^2}}$

$\lim_{x\to 3^+} \frac{4(x-3)}{|x-3|} = 4$

E. $\lim_{x\to 0}\left[\frac{\frac{1}{x}+\frac{1}{x+3}}{x}\right]\left[\frac{x(x+3)}{x(x+3)}\right]$

$\lim_{x\to 0} \frac{x+3+x}{x^2(x+3)}$

$\lim_{x\to 0} \frac{2x+3}{x^2(x+3)} = \frac{3}{0}$

$\therefore \lim_{x\to 0} f(x)$ d.n.e.

F. $\lim_{h\to 0} \frac{(x+h)^2-3(x+h)-(x^2-3x)}{h}$

$\lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{3x}-3h-\cancel{x^2}+\cancel{3x}}{h}$

$\lim_{h\to 0} \frac{\cancel{h}(2x+h-3)}{\cancel{h}}$

$\lim_{h\to 0} (2x+h-3) = 2x-3$

Topic 1.8: Determining Limits Using the Squeeze Theorem

The Squeeze Theorem is a technique used to confirm the limit of a function by comparison with two other functions whose limits are known or easily computed. Consider some function $f(x)$ is "trapped between two functions" on an interval containing point c. Let $f, g,$ and h be functions defined on the interval except possibly at c itself. Then for $x \neq c$, in the interval $f(x) \le g(x) \le h(x)$ and, also that $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} g(x) = L$

EX #7: The Squeeze Theorem

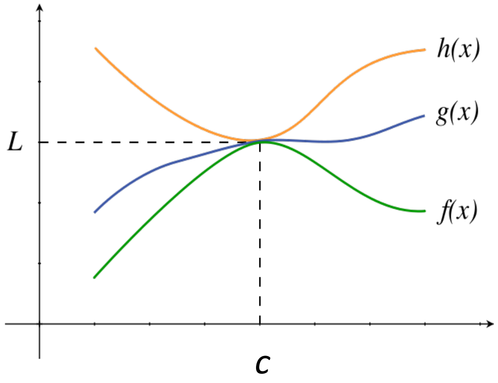


Photo: Public Domain

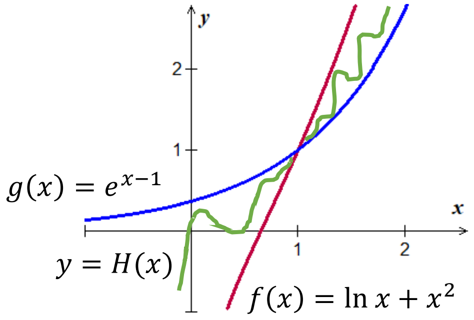
A. If $1 \le f(x) \le x^2+2x+2$, find $\lim_{x\to -1} f(x)$

$\lim_{x\to -1} 1 = 1$ $\lim_{x\to -1} (x^2+2x+2) = 1$

Since $\lim_{x\to -1} 1 = \lim_{x\to -1} (x^2+2x+2)$

Then, by the Squeeze Theorem $\lim_{x\to -1} f(x) = 1$.

B. Suppose $f(x)$ and $g(x)$ are boundaries of $H(x)$, as shown on the graph below. Given $f(x) = \ln x + x^2$ and $g(x) = e^{x-1}$, for all x in the interval containing $x = 1$, except possibly at $x = 1$ itself, find $\lim_{x\to 1} H(x)$. Justify.



Notice the limit at $x=1$ on $f(x)$ and $g(x)$ are both 1; but $f(x)$ and $g(x)$ change at $x=1$.

1) $\lim_{x\to 1} f(x) = 1$; $\lim_{x\to 1} g(x) = 1$

2) For $x<1$; $f(x) \leq H(x) \leq g(x)$

Since $\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^-} g(x)$

Then, $\lim_{x\to 1^-} H(x) = 1$

* left-hand limit $f(x) \leq g(x)$

3) For $x>1$; $g(x) \leq H(x) \leq f(x)$

Since $\lim_{x\to 1^+} g(x) = 1 = \lim_{x\to 1^+} f(x)$

Then, $\lim_{x\to 1^+} H(x) = 1$

* right-hand limit $g(x) \leq f(x)$

4) $\therefore \lim_{x\to 1^-} H(x) = \lim_{x\to 1^+} H(x) = 1$, by the Squeeze Theorem $\lim_{x\to 1} H(x) = 1$.

Lesson 6: Limits of Transcendental Functions

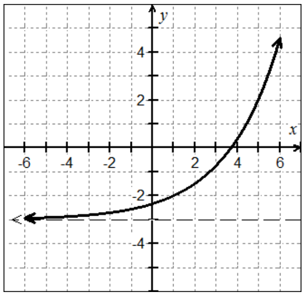
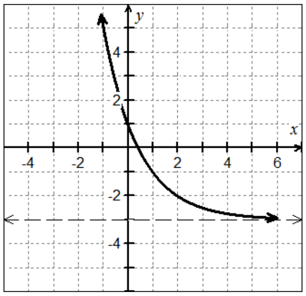
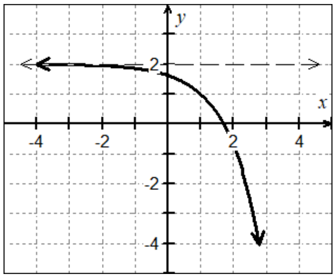
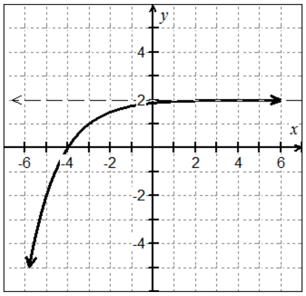
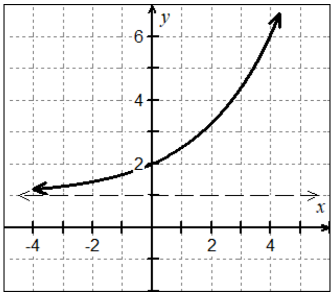
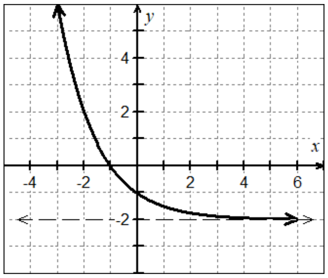
If you research the definition of a **transcendental function** you will find this explanation on Wolfram's Math World:

> *A function which is not an algebraic function. In other words, a function which "transcends," i.e., cannot be expressed in terms of, algebra. Examples of transcendental functions include the exponential function, the trigonometric functions, and the inverse functions of both.*

Topic 1.9: Connecting Multiple Representations of Limits

Analyzing Limits of Exponential Functions

EX #1: Recall that exponential equations are written in the form $y = ab^{(x-h)} + k$. You will need to find limits of exponential functions without the aid of a graph or calculator in this course. Do you remember the rules for transformations of exponential functions? Evaluate the limits using the graphs and look for patterns.

Exponential Growth	Exponential Decay
A. $f(x) = \left(\frac{3}{2}\right)^{x-1} - 3$ $\lim_{x\to-\infty} f(x) = -3$ $\lim_{x\to\infty} f(x) = \infty$ increasing  	**D.** $f(x) = \left(\frac{1}{2}\right)^{x-2} - 3$ $\lim_{x\to-\infty} f(x) = \infty$ $\lim_{x\to\infty} f(x) = -3$ decreasing 
B. $f(x) = -e^{x-1} + 2$ $\lim_{x\to-\infty} f(x) = 2$ $\lim_{x\to\infty} f(x) = -\infty$ decreasing  	**E.** $f(x) = -\left(\frac{1}{2}\right)^{x+3} + 2$ $\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to\infty} f(x) = 2$ increasing 
C. $f(x) = \left(\frac{2}{3}\right)^{-x} + 1$ $\lim_{x\to-\infty} f(x) = 1$ $\lim_{x\to\infty} f(x) = \infty$ increasing  	**F.** $f(x) = 2^{-x} - 2$ $\lim_{x\to-\infty} f(x) = \infty$ $\lim_{x\to\infty} f(x) = -2$ decreasing 

EX #2: WHAT JUST HAPPENED? Did you see that? Basically, the end-behavior of any exponential function tends toward three places.

CASE #1:	**CASE #2:**	**CASE #3:**
$-\infty$	∞	$y=b$

In PreCalculus, we learned the following rules without considering the behavior for negative *a*-values.

$\boldsymbol{y = ab^{(x-h)} + k}$	
$\boldsymbol{a > 0}$ **and** $\boldsymbol{b > 1}$	$\boldsymbol{a > 0}$ **and** $\boldsymbol{0 < b < 1}$
f is increasing b is exponential growth factor	f is decreasing b is exponential decay factor

Growth and decay are not synonymous with increasing and decreasing.

Consider the behavior of the function in relation to the horizontal asymptote. Notice whether the function is moving away from or toward the horizontal asymptote.

Also, what effect does the negative coefficient on *x* do to the function behavior?

EX #3: Now, use the equations from example #1 to complete the table below:

Graph of $f(x)$	Base	Increase or Decrease	$\lim\limits_{x\to-\infty} f(x)$	$\lim\limits_{x\to\infty} f(x)$
GROWTH	$b > 1$			
$f(x) = \left(\frac{3}{2}\right)^{x-1} - 3$	$\frac{3}{2}$	Increase	-3	∞
$f(x) = -e^{x-1} + 2$	e	$a=-1$ decrease	2	$-\infty$
$f(x) = \left(\frac{2}{3}\right)^{-x} + 1$	$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$	increase	1	∞
DECAY	$0 < b < 1$			
$f(x) = \left(\frac{1}{2}\right)^{x-2} - 3$	$\frac{1}{2}$	decrease	∞	-3
$f(x) = -\left(\frac{1}{2}\right)^{x+3} + 2$	$\frac{1}{2}$	$a=-1$ increase	$-\infty$	2
$f(x) = 2^{-x} - 2$	$2^{-1} = \frac{1}{2}$	decrease	∞	-2

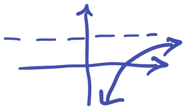
EX #4: You got this! Find the limits of each of the following exponential functions.

A. $\lim_{x\to-\infty} -(0.3)^x + 2 = -\infty$

b < 1 decay

a < 0 reflects

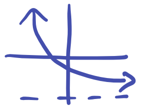
increasing



B. $\lim_{x\to-\infty} e^{-x+1} - 3 = \infty$

$\frac{1}{e} < 1$ decay

decreasing

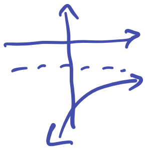


C. $\lim_{x\to\infty} -3^{-x+1} - 3 = -3$

3^{-1} decay

a < 0 reflects

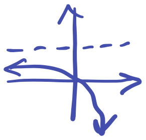
increasing



D. $\lim_{x\to\infty} -\left(\frac{1}{3}\right)^{-x} + 1 = -\infty$

$\left(\frac{1}{3}\right)^{-1} > 1$ growth

a < 0 decreasing



E. $\lim_{x\to2}(e^{x-3} - 1) = \frac{1-e}{e}$

e > 1

growth

substitution

F. $\lim_{x\to-3}\left[\left(\frac{1}{2}\right)^{-x-1} + 5\right] = \frac{21}{4}$

$\left(\frac{1}{2}\right)^2 + 5$

$5\frac{1}{4}$

substitution

Analyzing Limits of Trigonometric Functions

Most of the work you have done with limits to this point have dealt with polynomial or rational functions. When confronted with trigonometric functions, you will find throughout the course, that there are several methods to use. Let's begin with the basics in this section.

Topic 1.7: Selecting Procedures for Determining Limits

EX#5: Use direct substitution to find each limit.

A. $\lim_{\theta\to\frac{\pi}{3}} \frac{\cos 2\theta}{\theta}$

$$\frac{\cos\left(\frac{2\pi}{3}\right)}{\frac{\pi}{3}} = \frac{-1/2}{\pi/3} \Rightarrow \frac{-3}{2\pi}$$

B. $\lim_{\theta\to\frac{\pi}{2}} 3\sin^2\theta$

$3\left(\sin\frac{\pi}{2}\right)^2$

$3(1) = 3$

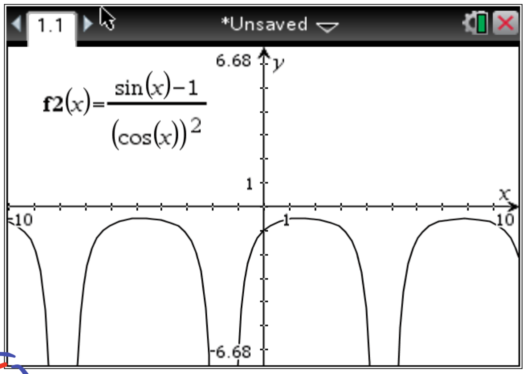
Just like some polynomial functions where a function value is not defined, yet a limit will exist…the same will occur with trigonometric functions.

EX #6: Evaluate the limit by direct substitution. $\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta - 1}{\cos^2\theta}$

$\lim_{\theta\to\frac{\pi}{2}}[\sin\theta - 1] = 0$ and $\lim_{\theta\to\pi/2}(\cos^2\theta) = 0$ indeterminate form

A few thoughts to consider:

If you algebraically determine that a function is undefined at a value, does that mean the limit does not exist? Can you look at the graph? ***How do you analyze the limit when no calculator or graph is permitted?*** We need to use some trig identities and rewrite the function. Try that here:



$$f2(x)=\frac{\sin(x)-1}{(\cos(x))^2}$$

$$\frac{\sin x - 1}{1-\sin^2 x} = \frac{-1\cdot(\cancel{\sin x - 1})}{(1+\sin x)\cancel{(1-\sin x)}}$$

$$f(x)=\frac{-1}{1+\sin x}$$

Topic 1.6: Determining Limits Using Algebraic Manipulation

EX #7: You Got This! Use your new skills to evaluate each limit below by first rewriting the function and using identities.

A. $\lim\limits_{\theta\to\frac{\pi}{2}} 2\cos\theta\cot\theta$

$$\lim_{\theta\to\pi/2} 2\cos\theta\left[\frac{\cos\theta}{\sin\theta}\right]$$

$$=\frac{2\left(\cos\frac{\pi}{2}\right)^2}{\sin \pi/2}$$

$$=\frac{0}{1}=\underline{\underline{0}}$$

B. $\lim\limits_{\theta\to\frac{\pi}{4}} \frac{\sin\theta\cos\theta}{\tan\theta}$

$$\lim_{\theta\to\pi/4}\frac{\sin\theta\cos\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$\lim_{\theta\to\pi/4}(\cos^2\theta)$$

$$=\left(\frac{\sqrt{2}}{2}\right)^2=\underline{\underline{\frac{1}{2}}}$$

C. $\lim\limits_{\theta\to\pi} \frac{\sin\theta\csc\theta}{3\theta}$

$$\lim_{\theta\to\pi}\frac{1}{3\theta}$$

$$\underline{\underline{\frac{1}{3\pi}}}$$

HAVE YOU DISCOVERED A PATTERN IN ALL OF THIS YET? If you ALWAYS try direct substitution first, three things will occur!

number value	$\frac{\text{number}}{0}$ *	$\frac{0}{0}$ *

factor, cancel, rationalize, analyze

* indeterminate form $\frac{0}{0}$; inspect one-sided limits $\begin{cases}\infty\\-\infty\\\text{dne}\end{cases}$

Later in the course you will learn a smooth technique known as L'Hospital's Rule. It will be a quicker method for evaluating many functions. For now, you might like to memorize these special rules, as well.

SPECIAL TRIGONOMETRIC LIMITS

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1 \qquad \lim_{\theta\to0}\frac{1-\cos\theta}{\theta}=0$$

EX #8: Use a graphing calculator and the table feature to evaluate the following trigonometric limits.

A. $\lim_{\theta\to0}\frac{\sin2\theta}{\theta}=2$	−0.1	−0.01	0	0.01	0.1
	1.998	1.999	dne	1.999	1.998
B. $\lim_{\theta\to0}\frac{\cos3\theta-1}{\theta}=0$	−0.1	−0.01	0	0.01	0.1
	0.446	0.044	dne	−0.044	−0.446
C. $\lim_{\theta\to0}\frac{\sin3\theta}{4\theta}=\frac{3}{4}$	−0.1	−0.01	0	0.01	0.1
	0.738	0.749	dne	0.749	0.738
D. $\lim_{\theta\to0}\frac{1-\cos3\theta}{2\theta}=0$	−0.1	−0.01	0	0.01	0.1
	−0.223	−0.022	dne	0.022	0.223

EX#9: Some algebraic tricks with properties.

A. $\lim_{\theta\to0}\frac{3\sin5\theta}{2\theta}$

$$\lim_{\theta\to0}\left[\frac{3\sin5\theta}{2\theta}\right]\left[\frac{5}{5}\right]$$

$$\lim_{\theta\to0}\left(\frac{15}{2}\right)\left(\cancelto{1}{\frac{\sin5\theta}{5\theta}}\right)$$

$$\underline{\underline{\frac{15}{2}}}$$

B. $\lim_{\theta\to0}\frac{1-\cos\theta+\sin3\theta}{\theta}$

$$\lim_{\theta\to0}\left(\cancelto{0}{\frac{1-\cos\theta}{\theta}}\right)+\lim_{\theta\to0}\left(\frac{\sin3\theta}{\theta}\right)\left(\frac{3}{3}\right)$$

$$=0+\lim_{\theta\to0}3\left[\cancelto{1}{\frac{\sin3\theta}{3\theta}}\right]$$

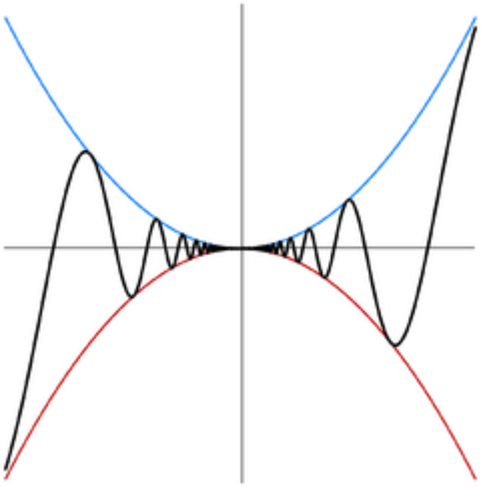
$$=0+3$$

$$=\underline{3}$$

Topic 1.8: Determining Limits Using the Squeeze Theorem

EX #10: Evaluate the limit using the squeeze theorem.

$$\lim_{x \to 0} \left[(x^2) \sin\left(\frac{1}{x}\right) \right]$$



A. Explain why you cannot use the Product Limit Law.

$\lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ can't be used because $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

B. Consider the range of the sine function $-1 \le \sin x \le 1$. We can conclude that $-1 \le \sin\left(\frac{1}{x}\right) \le 1$. Multiply through by x^2 to use the Squeeze Theorem.

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} (-x^2) = 0$

Since $\lim_{x \to 0} x^2 = \lim_{x \to 0} (-x^2)$ then by the Squeeze Theorem $\lim_{x \to 0} \left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0.$

EX #11: Find the limit. $\lim_{x \to 0} \left[(x^2) \cos\left(\frac{1}{x^2}\right) \right]$

Know:
$-1 \le \cos\theta \le 1$

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$$

$$-x^2 \le x^2 \cdot \cos\left(\frac{1}{x^2}\right) \le x^2$$

$$\lim_{x \to 0} (-x^2) \le \lim_{x \to 0} x^2 \cdot \cos\left(\frac{1}{x^2}\right) \le \lim_{x \to 0} (x^2)$$

$$0 \le \lim_{x \to 0} x^2 \cdot \cos\left(\frac{1}{x^2}\right) \le 0$$

$\therefore$ by Squeeze Theorem $\lim_{x \to 0} \left[x^2 \cdot \cos\left(\frac{1}{x^2}\right)\right] = 0$

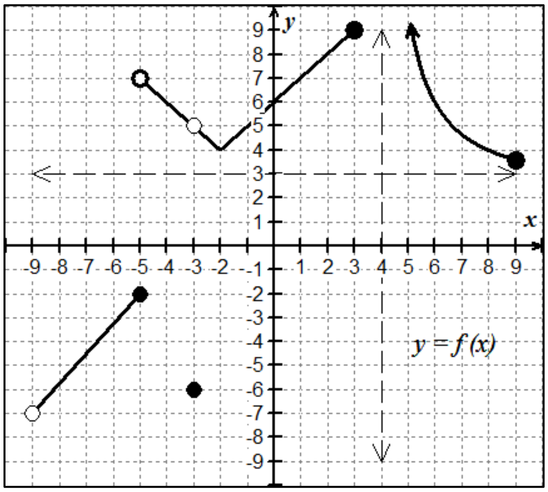
Lesson 7: Limits and Continuity

Now that we have an understanding of limits and using limit notation, we can use limits to define continuity at a point. ***This will be a very important concept throughout our course.***

Topic 1.11: Defining Continuity at a Point

EX #1: A Discovery Exploration.

Use the graph below to complete the table. You should ***look for three conditions that are necessary to satisfy the definition of continuity. That is, what three conditions must exist in order for $f(x)$ to be continuous at a point $x = c$?***



Definition of Continuity
$f(c)$ is defined
$\lim_{x\to c} f(x)$ exists
$\lim_{x\to c} f(x) = f(c)$

$x = c$	Find the function value, if it exists, for given x-value	Find left-hand limit $\lim_{x\to c^-} F(x)$	Find right-hand limit $\lim_{x\to c^+} F(x)$	Find general limit $\lim_{x\to c} F(x)$	Is $F(x)$ continuous at $x = c$?
$x = 9$ endpoint	3.5	3.5	dne	dne	one-sided yes
$x = 4$	dne	dne	∞	dne	no
$x = 3$	9	9	dne	dne	no
$x = 0$	6	6	6	6	yes
$x = -3$	-6	5	5	5	no
$x = -5$	-2	-2	7	dne	no

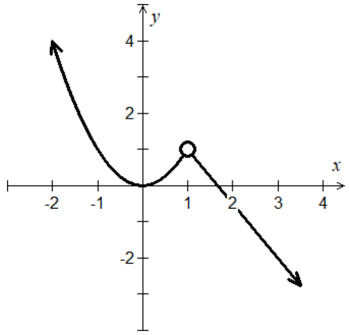
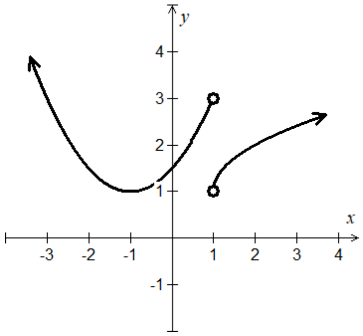
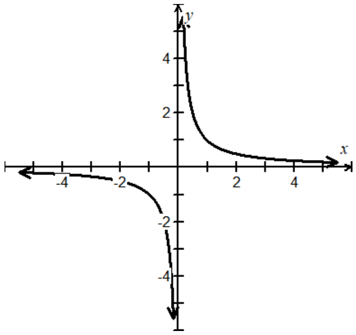
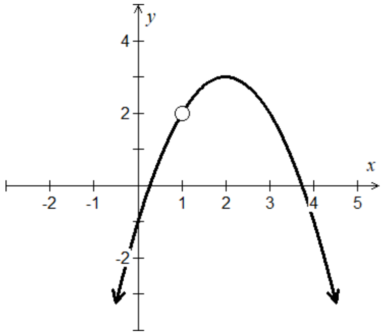
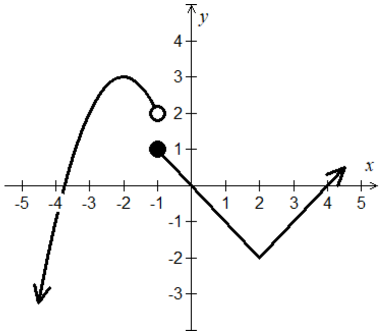
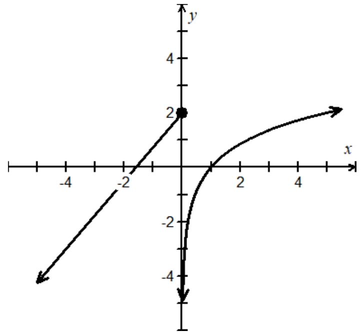
Analyze this table and write the three conditions for continuity above. Memorize the rules!

*While **limits** told us where a function intended to go.*

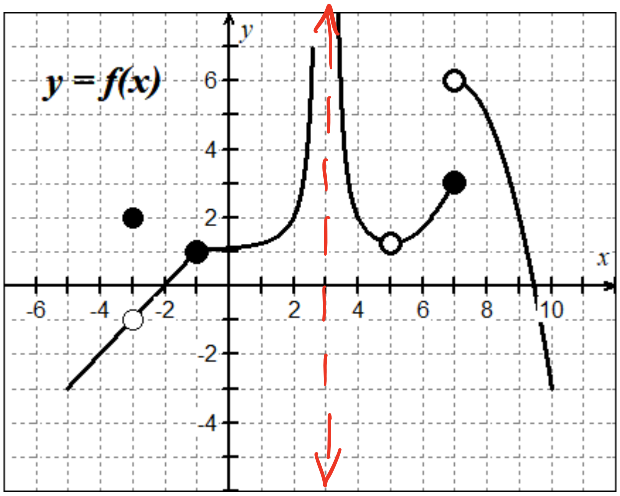
***Continuity** guarantees that the function actually made it there.*

Classifying Discontinuities

Topic 1.10: Exploring Types of Discontinuities

Removable or Point (Holes) **2-sided limit exists**	**Non-Removable**	
	Jump 1-sided limits exists	**Infinite** At least one of the 1-sided limits doesn't exist
		
		

EX #2: Find the points (intervals) at which the function is continuous, and the points at which the function is discontinuous on the interval $-5 < x < 10$.



continuous:

(−5,−3) (−3,3) (3,5) (5,7] (7,10)

discontinuities

x = −3 removable, hole

x = 3 vertical asymptote non-removable

x = 5 point discontinuity

x = 7 jump, non-removable

Definition of Continuity - More Facts and Theorems

One-Sided Continuity

A function $f(x)$ is called

- ☐ Left-continuous at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$
- ☐ Right-continuous at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$

Continuity at a Point

Suppose $f(x)$ is defined on an open interval containing $x = c$.
Then f is continuous at $x = c$ if

$$\lim_{x \to c} f\,(x) = f(c)$$

Continuity on an Open Interval

A function is continuous on an open interval $(a\,, b\,)$ if it is continuous at each point in the interval.

Continuity on a Closed Interval

A function is continuous on a closed interval $[a, b\,]$ if it is continuous on the open interval $(a\,, b\,)$ and the function is continuous from the right at a and continuous from the left at b.

Continuity Laws of Some Basic Functions

- ☐ Polynomial functions $P(x)$ are continuous over reals
- ☐ Rational Functions $P(x)/Q(x)$ is continuous on its domain such that $Q(c) \neq 0$.
- ☐ $y = x^{1/n}$ is continuous on all reals if n is odd and continuous on $[0, \infty)$ if n is even.
- ☐ $y = \sin x$ and $y = \cos x$ are continuous over reals
- ☐ $y = b^x$ is continuous for $b > 0, b \neq 1$
- ☐ $y = \log_b x$ is continuous for $x > 0, b > 0, b \neq 1$
- ☐ Inverse functions - if $f(x)$ is continuous on an interval with range R and $f^{-1}(x)$ exists, then $f^{-1}(x)$ is continuous on domain R.

Properties of Continuity

Given functions f and g continuous at $x = c$, then the following functions are also continuous at $x = c$.

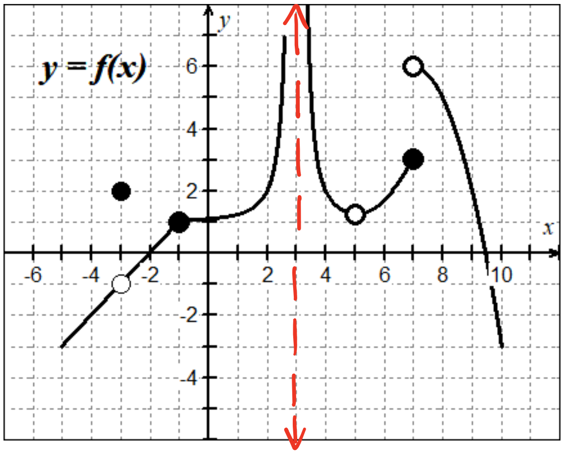
1. Scalar multiple: $b \cdot f$
2. Sum or difference: $f \pm g$
3. Product: $f \cdot g$
4. Quotient: $\frac{f}{g}$; if $g(c) \neq 0$
5. Compositions: If g is continuous at c and f is continuous at $g\,(c\,)$, then the composite function is continuous at c, $(f \circ g)(x) = f(g(x))$

EX #3: For $c = -3, c = 3,$ and $c = 7$, find $f(c)$, $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$, and $\lim_{x\to c} f(x)$. Justify your findings using the three-part definition of continuity.

$\underline{c=-3}$: $\lim_{x\to -3^-} f(x) = -1 = \lim_{x\to -3^+} f(x)$

$f(-3) = 2$

@ $x=-3$, f is not continuous b/c $\lim_{x\to -3} f(x) \neq f(-3)$



$\underline{c=3}$: $\lim_{x\to 3^-} f(x) = \infty = \lim_{x\to 3^+} f(x)$

$f(3)$ undefined

@ $x=3$, f is not continuous b/c $\lim_{x\to c} f(x) \neq f(3)$

$\underline{c=7}$: $\lim_{x\to 7^-} f(x) = 3$ $\lim_{x\to 7^+} f(x) = 6$, $f(7) = 3$

@ $x=7$, f is not continuous b/c $\lim_{x\to 7} f(x) \neq f(7)$

Topic 1.12: Confirming Continuity over an Interval

You must be able to confirm continuity without a graph or a calculator by using your knowledge of function behavior for parent functions and their transformations.

EX #4: Find the values of *x* where the function is discontinuous. Describe the type, infinite or removable. Justify your answer by using the definition of continuity.

A. $f(x) = \dfrac{1}{x-3}$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to 3^+} f(x) = \infty$

$f(3)$ undefined

f is not continuous

$x=3$ is vertical asymptote

infinite discontinuity

B. $g(x) = \dfrac{2x^2+7x+6}{x+2} \rightarrow \dfrac{(2x+3)\cancel{(x+2)}}{\cancel{(x+2)}}$

$\lim_{x\to -2^-} g(x) = -1$

$\lim_{x\to 2^+} g(x) = -1$

$g(-2)$ undefined

hole $(-2,-1)$

$\lim_{x\to -2} g(x) \neq g(-2)$

removable discontinuity at $x=-2$

not continuous

Topic 1.13: Removing Discontinuities

EX #5: Use the definition of continuity to find the value of k so that the function is continuous for all real numbers.

A. $g(x) = \begin{cases} kx^2, & x \le 2 \\ kx - 6, & x > 2 \end{cases}$

$\lim_{x\to 2^-} g(x) = 4k$

$\lim_{x\to 2^+} g(x) = 2k-6$

$\lim_{x\to 2} g(x)$: $4k = 2k-6$
$k = -3$

$\lim_{x\to 2} g(x) = -12$ and $g(x)$ is continuous at $x = 2$ if $k = -3$; $g(2) = \lim_{x\to 2} g(x) = -12$

B. $h(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x < 4 \\ 5k - 4x, & x \ge 4 \end{cases}$

$\lim_{x\to 4^-} h(x) = -1$

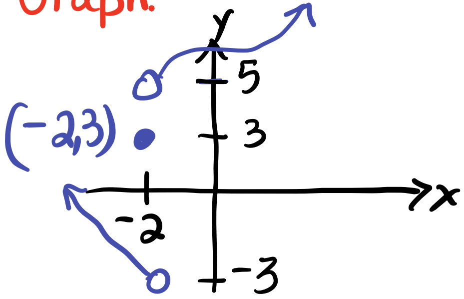
$\lim_{x\to 4^+} h(x) = 5k-16$

$\lim_{x\to 4} h(x)$: $5k-16 = -1$
$k = 3$

$\lim_{x\to 4} h(x) = -1$ and $h(x)$ is continuous at $x = 4$ if $k = 3$; $h(4) = \lim_{x\to 4} h(x) = -1$

EX #6: Given $h(x) = \begin{cases} -2x - 5\ ; & x < -2 \\ 3\ ; & x = -2 \\ x^3 - 6x + 3; & x > -2 \end{cases}$ for what values of x is $h(x)$ not continuous? Justify.

Graph:



Each "piece" is continuous, polynomials are continuous everywhere.

Investigate behavior at $x = -2$

Justify: 1) $h(-2) = 3$
2) $\lim_{x\to -2} h(x)$ dne
3) $\lim_{x\to -2} h(x) \ne h(-2)$

$\lim_{x\to -2^-} h(x) = -3$

$\lim_{x\to -2^+} h(x) = 5$

Conclusion: $h(x)$ is not continuous at $x = -2$

EX #7: Use the three-part definition of continuity to create a system of equations. Then, find the values of a and b so that $f(x)$ is continuous for all real numbers.

$$f(x) = \begin{cases} 2ax - b\ ; & x < -1 \\ 6\ ; & x = -1 \\ ax^2 + b; & x > -1 \end{cases} \Rightarrow \begin{matrix} -2a - b = 6 \\ a + b = 6 \end{matrix}$$

1)
$$\begin{aligned} a + b &= 6 \\ -2a - b &= 6 \\ \hline -a &= 12 \\ a &= -12 \end{aligned}$$

2)
$$\begin{aligned} a + b &= 6 \\ -12 + b &= 6 \\ b &= 18 \end{aligned}$$

$$\lim_{x \to -1^-} f(x) = 6$$

$$\lim_{x \to -1^+} f(x) = 6$$

$$f(-1) = 6$$

$$\therefore \begin{matrix} a = -12 \\ \underline{\underline{b = 18}} \end{matrix}$$

You know that the function value and the limit value can exist independently of each other. Let's summarize the big ideas of continuity on intervals, at points, and one-sided limits.

EX #8: Given $g(x) = \begin{cases} 2x + 1, & x < 3 \\ x^2, & x \geq 3 \end{cases}$ find each of the following.

A. $\lim\limits_{x \to 3^-} g(x)$

$2(3)+1$

$$\lim_{x \to 3^-} g(x) = 7$$

B. $\lim\limits_{x \to 3^+} g(x)$

$$\lim_{x \to 3^+} g(x) = 9$$

C. $g(3)$

$$g(3) = 9$$

D. $\lim\limits_{x \to 3} g(x)$

$\lim\limits_{x \to 3} g(x)$ does not exist

F. Is $g(x)$ continuous at $x = 3$? Justify.

$g(x)$ is not continuous b/c $\lim\limits_{x \to 3^-} g(x) \neq \lim\limits_{x \to 3^+} g(x)$ and $\lim\limits_{x \to 3} g(x) \neq g(3)$

Summary:

If a function is continuous at a point, then the function value and the limit value are the same at that point!

Lesson 8: Infinite Limits and Limits at Infinity

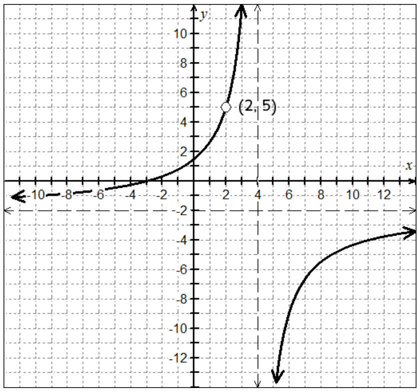
In our first lesson on *Understanding Limits* you were confronted with these two situations. Here we begin to **compare and contrast** the behavior of functions as they ***approach infinity***, as well as, functions that ***tend toward infinity*** in certain circumstances. Let's go…

Topic 1.14: Connecting Infinite Limits and Vertical Asymptotes

Topic 1.15: Connecting Limits at Infinity and Horizontal Asymptotes

EX #1: Use the graphs of $f(x)$ and $g(x)$ shown below to compare and contrast behaviors involving infinity. Then, write your discovery and definition.

$$f(x) = \frac{-2x^2 - 2x + 12}{x^2 - 6x + 8}$$

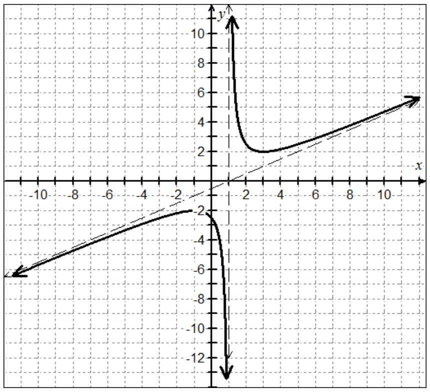


$\lim_{x\to 4^-} f(x) = \infty$	$\lim_{x\to 1^-} g(x) = -\infty$
$\lim_{x\to 4^+} f(x) = -\infty$	$\lim_{x\to 1^+} g(x) = \infty$

Your Discovery:
A vertical asymptote occurs if one-sided limits are unbounded.

An Infinite Limit is: As $x \to c$, the limit will approach either $+\infty$ or $-\infty$ (vertical asymptote)

$$g(x) = \frac{x^2 - 2x + 5}{2x - 2}$$



$\lim_{x\to -\infty} f(x) = -2$	$\lim_{x\to -\infty} g(x) = -\infty$
$\lim_{x\to \infty} f(x) = -2$	$\lim_{x\to \infty} g(x) = \infty$

Your Discovery:
As $x \to \pm\infty$, a horizontal or slant asymptote occurs

A Limit at Infinity is: As $x \to \pm\infty$, the limit will approach a y-value or will be unbounded (horizontal asymptote)

In Pre-Calculus you learned some basic truths about rational functions.

1. When a factor cancelled from the denominator a hole occurred.
2. When a factor would not cancel from the denominator a vertical asymptote occurred.

EX #2: Use the previous equations to find the one-sided limits analytically.

A. $\lim\limits_{x\to4^-}\dfrac{-2x^2-2x+12}{x^2-6x+8}$

$$\lim_{x\to4^-}\frac{-2(x+3)\cancel{(x-2)}}{(x-4)\cancel{(x-2)}}$$

$$\lim_{x\to4^-}\frac{-2(x+3)}{(x-4)} \Rightarrow \underline{\underline{\infty}}$$

B. $\lim\limits_{x\to4^+}\dfrac{-2x^2-2x+12}{x^2-6x+8}$

$$\lim_{x\to4^+}\frac{-2(x+3)}{(x-4)} \Rightarrow \underline{\underline{-\infty}}$$

$x=4.001 \quad f(x)=\dfrac{(-)}{(+)}$

C. $\lim\limits_{x\to1^-}\dfrac{x^2-2x+5}{2x-2}$

$$\lim_{x\to1^-}\frac{x^2-2x+5}{2(x-1)} \Rightarrow \underline{\underline{-\infty}}$$

$x=0.999 \quad f(x)=\dfrac{(+)}{(-)}$

D. $\lim\limits_{x\to1^+}\dfrac{x^2-2x+5}{2x-2}$

$$\lim_{x\to1^+}\frac{x^2-2x+5}{2(x-1)} \Rightarrow \underline{\underline{+\infty}}$$

$x=1.001 \quad f(x)=\dfrac{(+)}{(+)}$

Definition and Justification of Vertical Asymptotes

Case #1:

$$h(c)=\frac{non-zero}{zero}$$

$x=c$ is: Vertical asymptote

Case #2:

$$h(c)=\frac{zero}{zero}$$

$x=c$ is: location of a hole

IN CALCULUS, YOU MUST USE NEW LANGUAGE IN ORDER TO JUSTIFY!

LIMIT DEFINITION (JUSTIFICATION) OF A VERTICAL ASYMPTOTE

If $\lim\limits_{x\to c^-}f(x)=\pm\infty$ or $\lim\limits_{x\to c^+}f(x)=\pm\infty$

then $x=c$ is a vertical asymptote.

Topic 1.13: Removing Discontinuities

EX #3: Use the function below to find any vertical asymptote(s) that exist. Justify your answer using limits.

$$h(x) = \frac{2x^2 + 9x - 5}{x^2 + 3x - 10} \Rightarrow \frac{(2x-1)\cancel{(x+5)}}{(x-2)\cancel{(x+5)}} \Rightarrow \frac{2x-1}{x-2}$$

at $x=2$, there is a vertical asymptote b/c

$\lim_{x \to 2^-} h(x) = -\infty$ and $\lim_{x \to 2^+} h(x) = +\infty$

x	1.999	2.001
$h(x)$	$(+)/(-)$	$(+)/(+)$

behavior

Limits at Infinity

Next, we will explore **limits at infinity** in order to differentiate between the two conditions. Recall the lessons from Pre-Calculus related to analyzing the **end behavior of functions**. In the exercise below, use this prior knowledge to find each limit at infinity.

EX #4: Find each limit at infinity, explain your thinking.

A. $\lim_{x \to \infty} (x^2 - 4)(x^2 + 3) = \infty$

This function is quartic, by end-behavior model the leading coefficient rises as $x \to \infty$ $f(x)$ increases without bound.

B. $\lim_{x \to -\infty} (5x^3 - 2x + 4) = -\infty$

A cubic function with a positive leading coefficent falls as $x \to -\infty$ $f(x)$ decreases without bound.

C. $\lim_{x \to \infty} \frac{3x^2 - 4}{x^2 + 1} = 3$

denominator power = numerator power

compare coefficients to find horizontal asymptote

as $x \to \infty$, $y \to 3$; $y=3$ H.A.

D. $\lim_{x \to -\infty} \frac{5x - 2}{x^2 + 1} = 0$

denominator power > numerator power

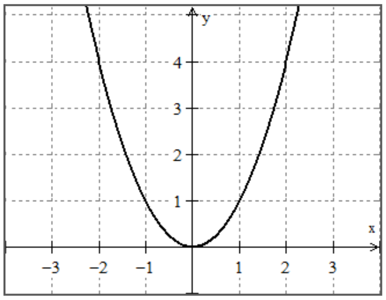
$y=0$ is horizontal asymptote.

as $x \to -\infty$, $y \to 0$; $y=0$ H.A.

EX #5: There are only four possible outcomes when you explore behavior to the extreme right or left.

A. The curve can increase without bound.

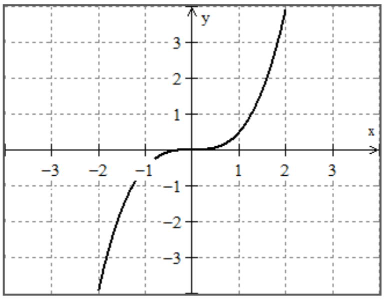
$$\lim_{x\to\infty} f(x) = \infty$$



$x \to \infty$
$y \to \infty$

B. The curve can decrease without bound.

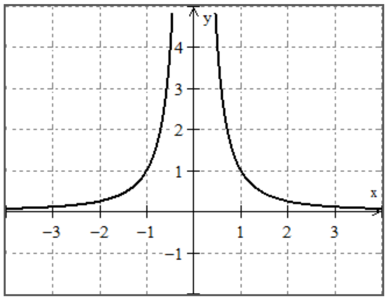
$$\lim_{x\to-\infty} f(x) = -\infty$$



$x \to -\infty$
$y \to -\infty$

C. The curve can become asymptotic to the *x*-axis.

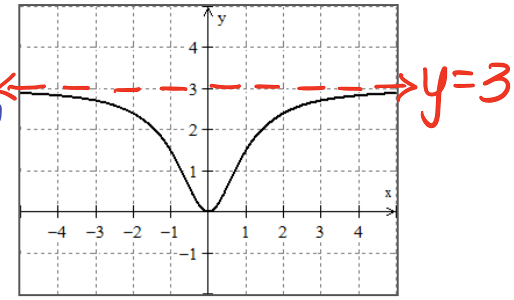
$$\lim_{x\to\infty} f(x) = 0$$



$x \to \infty$
$y \to 0$

D. The curve can become asymptotic to a specific *y*-value.

$$\lim_{x\to-\infty} f(x) = 3$$



$x \to -\infty$
$y \to 3$

Calculus Knowledge for Asymptotes

Revisiting the rules for finding potential horizontal asymptotes for rational functions from PreCalculus, you can use the idea of a limit and calculus to see why those rules hold true.

1. If degree of numerator is less than degree of denominator (bottom heavy), then limit is zero.

$$\lim_{x\to\pm\infty} f(x) = 0$$

2. If degree of numerator equals degree of denominator (powers equal), then limit is the ratio of coefficients of the highest degree.

$$\lim_{x\to\pm\infty} f(x) = \frac{\text{coefficient of numerator's highest power}}{\text{coefficient of denominator's highest power}}$$

3. If degree of numerator is greater than degree of denominator (top heavy*), then limit does not exist.

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

Topic 1.9: Connecting Multiple Representations of Limits

EX #6: Divide every term in the rational expression by the ***highest power of x that appears in the denominator.*** Then, apply the Properties of Limits to evaluate each “piece” to find the limit at infinity, end behavior.

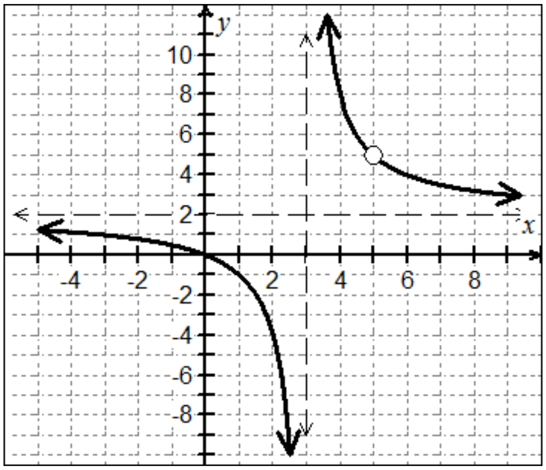
A. $\lim\limits_{x\to\infty} \dfrac{2x^2 - 10x}{x^2 - 8x + 15}$

divide each term by x^2

$$\lim_{x\to\infty} \frac{2-\frac{10}{x}}{1-\frac{8}{x}+\frac{15}{x^2}} \Rightarrow \frac{2-0}{1-0+0}$$

$\therefore$ as $x \to \infty$, $y \to 2$

$$\underline{\underline{\lim_{x\to\infty} f(x) = 2}}$$



horizontal asymptote

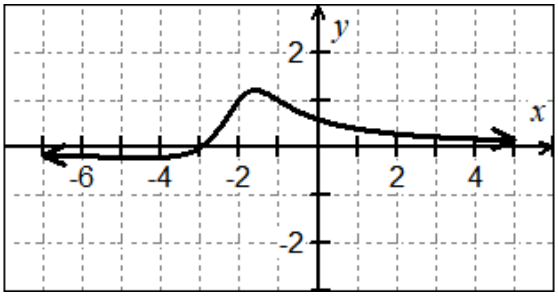
B. $\lim\limits_{x\to\infty} \dfrac{x + 3}{x^2 + 4x + 5}$

divide each term by x^2

$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{3}{x^2}}{1+\frac{4}{x}+\frac{5}{x^2}} \Rightarrow \frac{0+0}{1+0+0}$$

$\therefore$ as $x \to \infty$, $y \to 0$

$$\underline{\underline{\lim_{x\to\infty} f(x) = 0}}$$



horizontal asymptote

C. $\lim\limits_{x\to\infty} \dfrac{x^2 - 4x - 3}{x - 2}$

divide each term by x

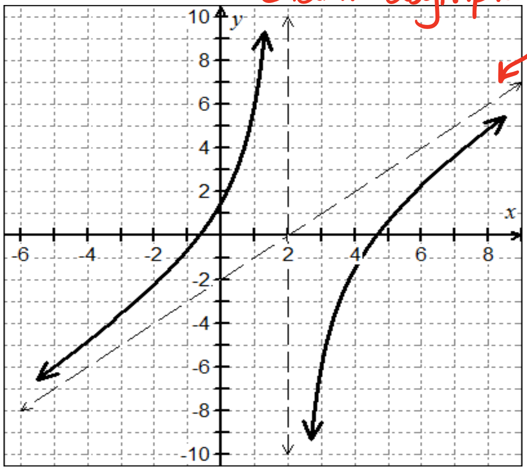
$$\lim_{x\to\infty} \frac{x-4-\frac{3}{x}}{1-\frac{2}{x}} \Rightarrow \infty$$

Slant Asymptote

$\underline{y = x - 2}$

$$\begin{array}{r|rrr} 2 & 1 & -4 & -3 \\ & \downarrow & 2 & -4 \\ \hline & 1 & -2 & \boxed{-7} \end{array}$$

slant asymptote



unbounded behavior

EX #7: Summarize and discuss characteristics and end behavior at horizontal asymptotes and slant asymptotes based on your observations in EX #6.

Functions with Horizontal Asymptotes

$\lim_{x\to\infty} f(x) = b$

$\lim_{x\to-\infty} f(x) = b$

$b \in \mathbb{R}$

The line $y=b$ is a horizontal asymptote of the graph of $y=f(x)$.

Functions with Slant Asymptotes

$\lim_{x\to\infty} f(x) = \pm\infty$

$\lim_{x\to-\infty} f(x) = \pm\infty$

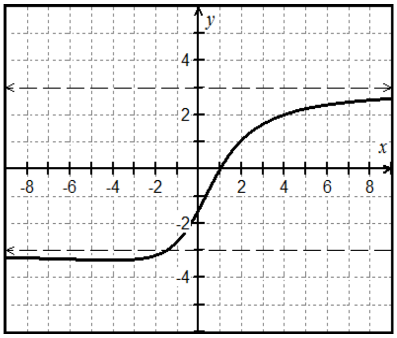
Use long division to find the equation for slant asymptote. TEST VALUES to find end-behavior.

IN CALCULUS, YOU MUST USE NEW LANGUAGE IN ORDER TO JUSTIFY!

LIMIT DEFINITION (JUSTIFICATION) OF A HORIZONTAL ASYMPTOTE

If $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$, then $y=b$ is a horizontal asymptote of $y=f(x)$.

EX #8: CHALLENGE! Use algebraic techniques to find the limits for $g(x) = \frac{3x-3}{\sqrt{x^2+4}}$, whose graph is shown.



$\sqrt{x^2+4} = |x|$ as $x\to\infty$ divide terms by $\sqrt{x^2}$

$$\lim_{x\to\infty} \frac{3-\frac{3}{|x|}}{\sqrt{1+\frac{4}{x^2}}} \Rightarrow \frac{3-0}{\sqrt{1+0}} = \underline{\underline{3}}$$

similarly as $x\to-\infty$

$$\lim_{x\to-\infty} \frac{-3-\frac{3}{x}}{\sqrt{1+\frac{4}{x^2}}} \Rightarrow \frac{-3-0}{\sqrt{1+0}} = \underline{\underline{-3}}$$

Since $\frac{3x}{|x|} = -3$ for $x\to-\infty$

$$\lim_{x\to-\infty} \frac{3x-3}{\sqrt{x^2+4}} = -3$$

$$\lim_{x\to\infty} \frac{3x-3}{\sqrt{x^2+4}} = 3$$

Lesson 9: Intermediate Value Theorem

Continuity of a function will prove to be an important characteristic in many theorems for our course. In order for theorems to be properly applied, we need to meet the certain conditions. Namely, the “if” part, called the ***hypothesis***, must be satisfied before we can apply the “then” part, or ***conclusion*** (consequence) of the theorem.

In this lesson, we will explore our first **existence theorem**. Existence theorems allow us to draw conclusions about a function’s behavior on an interval without precisely locating that behavior.

Topic 1.16: Working with the Intermediate Value Theorem (IVT)

Intermediate Value Theorem

The Big Idea:

The **IVT** says that a continuous function on an interval cannot skip values.

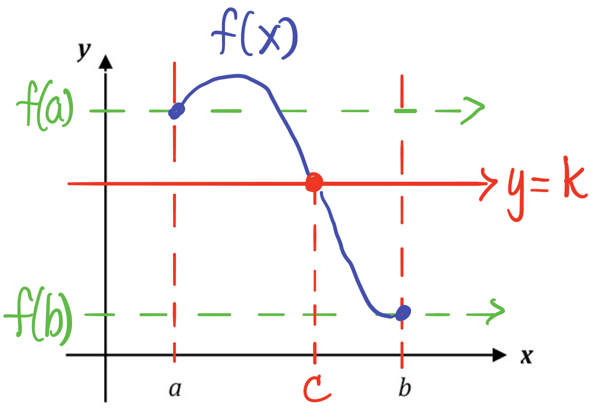
Using If/Then statements:

If a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, **Then** $f(x)$ takes on every value between $f(a)$ and $f(b)$ on that interval.

Another Approach:

Said in a different fashion, if we know a y-value, say $y = k$, that resides between the two endpoints, $f(a)$ and $f(b)$, then we are guaranteed at least one x- value, $x = c$, between the endpoints that generates that y-value, such that $f(c) = k$.

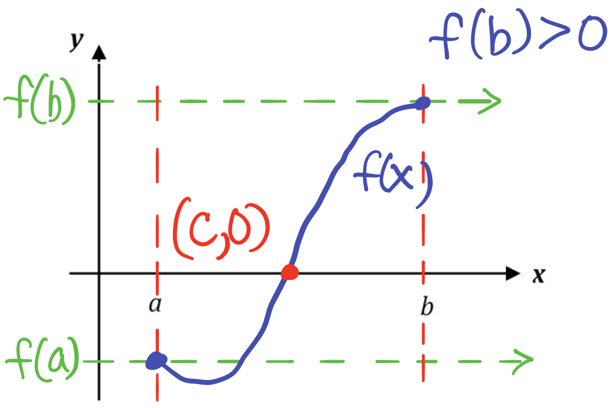
EX #1: Graphical representation of the Intermediate Value Theorem and its corollary:



$a < c < b$

$f(b) < c < f(a)$

A continuous function on an interval cannot skip values.

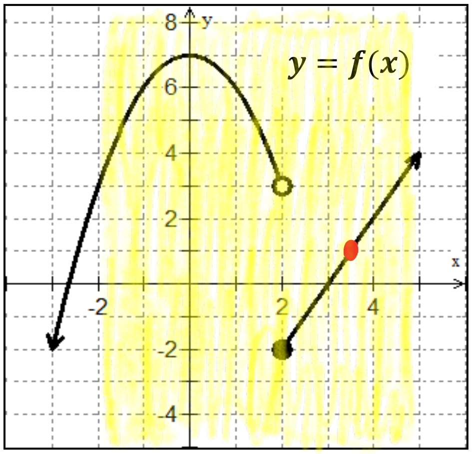


$f(a) < 0$

$f(a) < 0 < f(b)$

An important outcome of I.V.T. is that it can be helpful in finding zeros of a continuous function on an interval.

EX #2: Does the Intermediate Value Theorem apply on the specified intervals for the piecewise functions shown below? Be sure to explain completely.

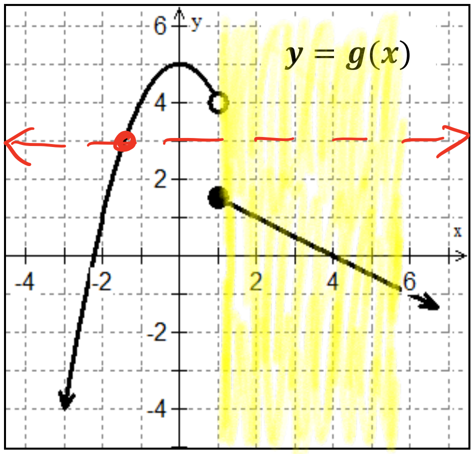


A. Is there a value of $x = c$ on the interval $[-2, 5]$ such that $f(c) = 1$?

yes, c exists on $-2 \leq x \leq 5$

B. Does I.V.T. guarantee a value of c on the interval $[-2, 5]$ such that $f(c) = 1$? Justify.

no, IVT does not guarantee a value of $x = c$ on $[-2, 5]$ since $f(x)$ is not continuous on the interval.



C. Is there a value of $x = c$ on the interval $[1, 6]$ such that $g(c) = 3$?

no, there is no value where $g(c) = 3$ on $1 \leq x \leq 6$.

D. Does I.V.T. guarantee a value of c on the interval $[1, 6]$ such that $g(c) = 3$? Justify.

no, IVT will not guarantee a value where $x = c$ on $[1, 6]$. $g(c) = 3$ does not occur on this interval.

What three conditions are necessary to apply the Intermediate Value Theorem?

1. f is continuous function on $[a, b]$
2. $f(a) \neq f(b)$
3. $f(c)$ must be between $f(a)$ and $f(b)$

EX #3: Apply the IVT, if possible, on [0, 5] so that $f(c) = 1$ for the function $f(x) = x^2 + x - 1$.

1) $f(x)$ is continuous over reals
2) $f(0) = -1$, $f(5) = 29$
3) Then, by IVT, there exists a value c, where $f(c) = 1$ since $f(x)$ takes on every value from $-1 < 1 < 29$ on $0 \leq c \leq 5$.

EX #4: Use the Intermediate Value Theorem to show that $f(x) = x^3 + 2x - 1$ has a zero in the interval [0, 1].

1) polynomials are continuous everywhere
2) $f(0) = -1$ and $f(1) = 2$; $f(0) \neq f(1)$
3) Since $f(0) < 0$ and $f(1) > 0$, there must be a value where $f(c) = 0$ on the interval $0 \leq x \leq 1$ by I.V.T.

EX #5: A car travels on a straight track. During the time interval $0 \leq t \leq 60$ seconds, the car's velocity v, measured in feet per second is a continuous function. The table below shows selected values of the function.

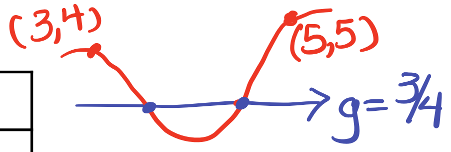
t, in seconds	0	15	25	30	35	50	60
$v(t)$ in ft/sec	−20	−30	−20	−14	−10	0	10

A. For $0 < t < 60$, must there be a time t when $v(t) = -5$?

B. Justify your answer.

1) $v(t)$ is continuous
2) $v(35) = -10$ and $v(50) = 0$
3) By IVT, there is a time, t, where $v(t) = -5$ is guaranteed on interval $35 \leq t \leq 50$ since $-10 < -5 < 0$.

EX #6: Let $g(x)$ be a continuous function. Selected values of $y = g(x)$ are given in the table below. For which value of k will the equation $g(x) = 3/4$ have **at least two solutions** on the closed interval $[2, 8]$?



x	2	3	4	5	8
$g(x)$	3	4	k	5	2

$K<3/4$

(A) 1 **(B)** $\frac{3}{4}$ [✓ **(C)** $\frac{9}{16}$] **(D)** $\frac{3}{2}$

$\frac{9}{16}<\frac{3}{4}$

EX #7: For the function $f(x) = \begin{cases}(x-3)^2, x = 5\\ 6, \quad 5 < x \le 10\end{cases}$. Find $f(5)$ and $f(10)$. Does IVT guarantee a y-value $f(c) = k$, on $5 \le x \le 10$ such that $f(5) < f(c) < f(10)$? Justify your answer.

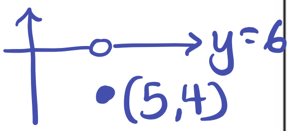
1) $f(5)=4$

2) $f(10)=6$

3) $\lim_{x\to 5^+} f(x)=6$

4) $\lim_{x\to 5^+} f(x)\neq f(5)$

5) Therefore, IVT does not apply since $f(x)$ is not continuous on $[5,10]$ at $x=5$. There is no guarantee for $x=k$ where $f(5)<k<f(10)$



EX #8: The functions f and g are continuous for all real numbers. The table below gives values of the functions at selected values of x The function h is given by $h(x) = g\big(f(x)\big) + 2$**.** Explain why there must be a value w for $1 < w < 6$ such that $h(w) = 0$.

x	1	2	6	8
$f(x)$	2	9	8	13
$g(x)$	3	-12	5	28

1) $f+g$ are continuous everywhere, so $h(x)$ is continuous on $[1,6]$.

2) $h(1)=g(f(1))+2$
$\quad = g(2)+2$
$h(1) = -10$

3) $h(6)=g(f(6))+2$
$\quad = g(8)+2$
$h(6) = 30$

4) By IVT, there exists a value $x=w$ where $h(w)=0$ since $h(1)<0<h(6)$ on the interal $1<w<6$.